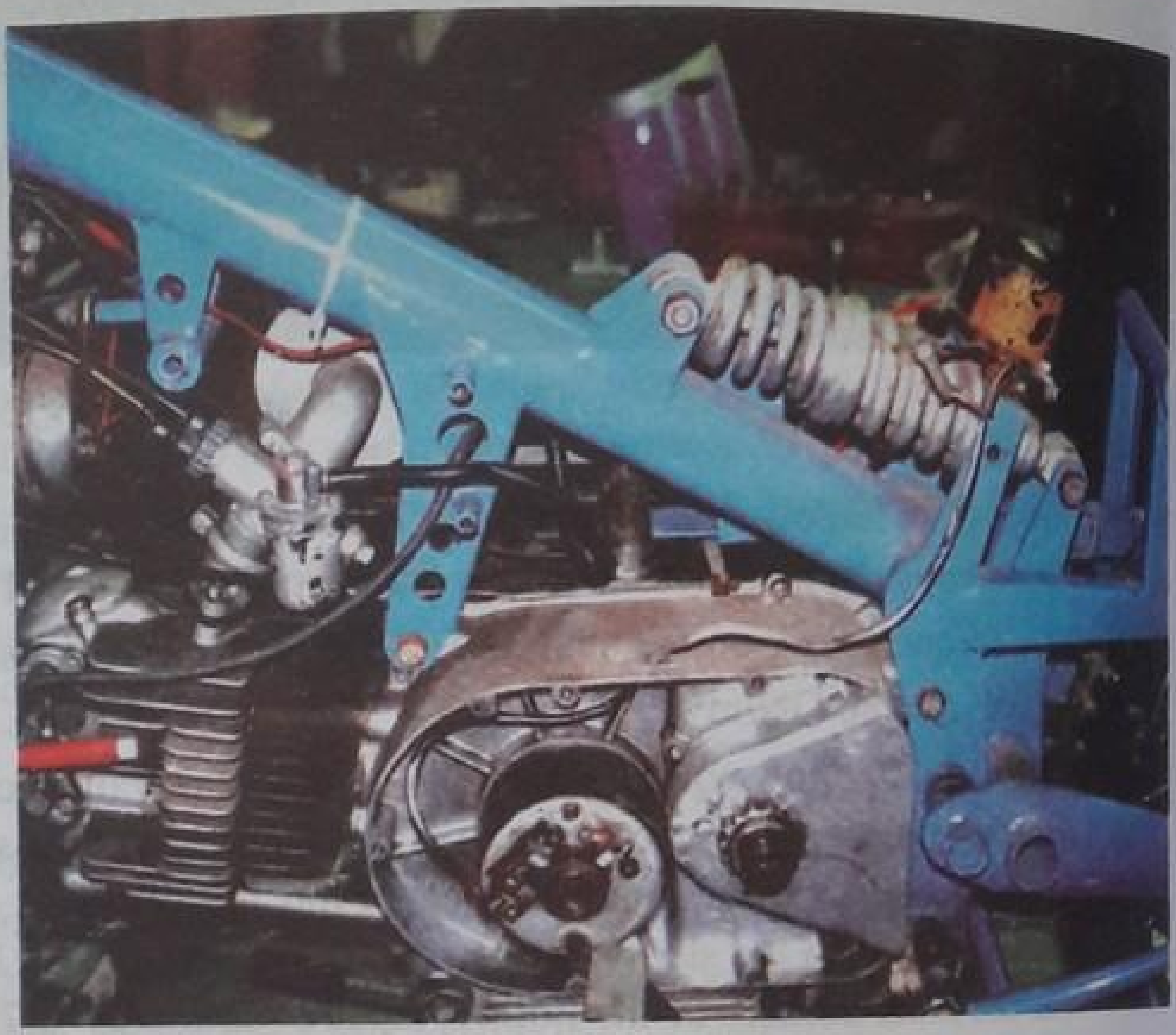
Chapter

14

Springs



14·1. INTRODUCTION

Springs are elastic members which distort under load and regain their original shape when load is removed. They are used in railway carriages, motor cars, scooters, motorcycles, rickshaws, governors etc. According to their uses, the springs perform the following **functions:**

(i) To absorb shock or impact loading as in *carriage springs*.

(ii) To store energy as in *clock springs*.

(iii) To apply forces to and to control motions as in *brakes and clutches*.

(iv) To measure forces as in *spring balances*.

(v) To change the variations characteristic of a member as in *flexible mounting of motors*.

The springs are usually made of either high carbon steel (0·7 to 1·0%) or medium carbon alloy steels. Phosphor bronze, brass, 18/8 stainless steel and monel and other metal alloys are used for corrosion resistance springs.

Various types of springs are employed for different purposes, some of them are as follows:

1. **Helical springs:**
   (*i*) Close-coiled helical springs; (*ii*) Open-coiled helical springs;
   (*iii*) Tension helical springs; (*iv*) Compression helical springs.
2. **Leaf springs:**
   (*i*) Full-elliptic; (*ii*) Semi-elliptic; (*iii*) Cantilever.
3. Torsion springs
4. Circular springs
5. Belleville springs
6. Flat springs.

14·2. HELICAL SPRINGS

A helical spring is a length of wire or bar wound into a helix. There are mainly two types of helical springs : (*i*) *Close-coiled*, and (*ii*) *open-coiled*.

14·3. THE CLOSE-COILED HELICAL SPRINGS

14·3·1. Close-coiled helical spring with 'Axial load'

A. Circular section wire springs:

In Fig. 14·1 is shown a close-coiled helical spring loaded with an axial load W.

Let, R = Radius of the coil,

d = Diameter of the wire of the coil,

δ = Deflection of coil under the load W,

C = Modulus of rigidity,

n = Number of coils or turns,

θ = Angle of twist,

l = Length of wire = $2\pi Rn$,

τ = Shear stress, and

I_p = Polar moment of inertia $= \dfrac{\pi}{32} d^4$.

It may be noted that each section of the coil is under torsion but there are small bending and shearing stresses which being small are usually neglected.

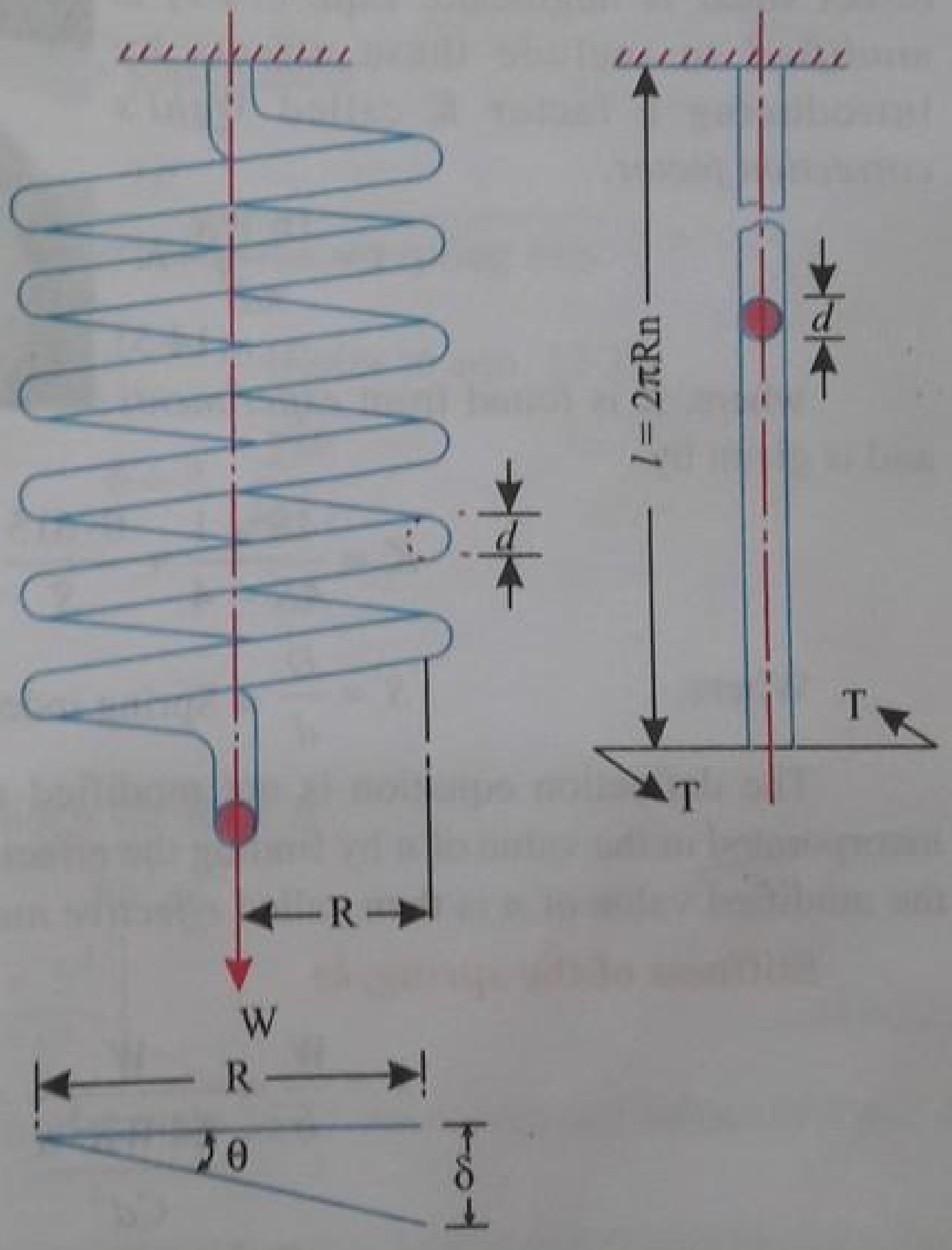


Fig. 14.1. Close-coiled helical spring.

Shear stress τ :

From torsion equation,

$$\frac{T}{I_p} = \frac{C\theta}{l} = \frac{\tau}{r}; \quad \frac{T}{I_p} = \frac{\tau}{r}$$

or, $$T = \frac{\tau I_p}{r} = \frac{\tau \times \pi d^4}{32} \times \frac{2}{d} = \tau \cdot \frac{\pi}{16} d^3$$

or, $$\tau = \frac{16T}{\pi d^3}$$

or, $$\tau = \frac{16\,WR}{\pi d^3} \quad (\because T = WR) \qquad ...(14.1)$$

Deflection, δ:

Again, $$\frac{T}{I_p} = \frac{C\theta}{l}$$

$$\theta = \frac{Tl}{CI_p} = \frac{WR \times 2\pi\, Rn \times 32}{C \times \pi d^4} = \frac{64\,WR^2\, n}{Cd^4} \qquad ...(14.2)$$

but, $$\delta = R \times \theta \qquad ...(14.3)$$

$\therefore$ $$\delta = \frac{64\,WR^3\, n}{Cd^4} \qquad ...(14.4)$$

Wahl's correction factor:

While deriving eqns. (14.1) and (14.2) the effect of curvature of spring and direct shear is neglected. Eqn. (14.1) is *modified* to include these effects by introducing a factor K called *Wahl's correction factor.*

$\therefore$ $$\tau = \frac{16\,WR}{\pi d^3} K \qquad ...(14.5)$$

where, K is found from *experiments* and is given by

Shock absorbers of automobiles have springs.

$$K = \frac{4S - 1}{4S - 4} + \frac{0.615}{S} \qquad ...(14.6)$$

Where, $S = \frac{D}{d}$ = Spring index (where, D = mean diameter of the coil)

The deflection equation is not modified as the effect, if any, is considered to have been incorporated in the value of n by finding the effect on deflection due to end coils experimentally and the modified value of n is then called *effective number* of coils.

Stiffness of the spring, k:

$$k = \frac{W}{\delta} = \frac{W}{\dfrac{64\,WR^3\, n}{Cd^4}} = \frac{Cd^4}{64R^3 n}$$

i.e. $$k = \frac{Cd^4}{64\,R^3 n} \qquad ...(14.7)$$

Energy stored, U:

Energy stored, $$U = \frac{1}{2} \times T \times \theta = \frac{1}{2} W \cdot R \times \frac{64\,WR^2\, n}{Cd^4}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \frac{16\,WR}{Cd^3} \cdot \frac{8\,WR^2\, n}{d} = \frac{1}{4C} \cdot \frac{16\,WR}{\pi d^3} \cdot \frac{16\,WR}{\pi d^3} \left[2\pi R\, nd^2 \times \frac{\pi}{4}\right]$$

$$= \frac{1}{4C} \cdot \tau^2 \times \text{volume of wire}$$

i.e. $$U = \frac{\tau^2}{4C} \times \text{volume of wire} \quad \text{...(14·8)}$$

Again, energy stored,

$$U = \frac{1}{2} \cdot T \cdot \theta = \frac{1}{2} \cdot W.R. \frac{\delta}{R} = \frac{1}{2} \cdot W \cdot \delta \quad (\because \delta = R\theta)$$

i.e. $$U = \frac{1}{2} W \delta \quad \text{...(14·9)}$$

B. Rectangular and square section wire springs:

Rectangular and square section wire springs are also used in many applications.

Here, $$\tau = \alpha \frac{T}{bh^2} \quad \text{...(14·10) (Refer to eqn. 13·38)}$$

Where, $$\alpha = 3 + 1 \cdot 8 \frac{h}{b}$$

[b = longer side, h = smaller side (For rectangular section)
$b = h$ and $\alpha = 4 \cdot 8$ ...(For square section)]

$\therefore$ $$\tau = \alpha \frac{WR}{bh^2} \cdot K \text{ with correction factor } K \quad \text{...(14·11)}$$

where, $$K = \frac{4S - 1}{4S - 4} + \frac{0 \cdot 615}{S} \quad (S = \text{spring index})$$

where, $$S = \frac{2R}{\text{Side of the section perpendicular to spring axis}}$$

Also, $$\delta = R\theta, \text{ where } \theta = \frac{\beta}{bh^3} \cdot \frac{Tl}{C} \quad \text{[Refer to eqn. 13·39]}$$

$\therefore$ $$\delta = \frac{R\beta}{bh^3} \cdot \frac{Tl}{C} \quad \text{or} \quad \delta = \beta \cdot \frac{TlR}{bh^3 C}$$

where, $$\beta = \frac{3 \cdot 5 (b^2 + h^2)}{b^2}$$

$$l = 2\pi Rn \; ; \; T = WR$$

$\therefore$ $$\delta = \frac{3 \cdot 5 (b^2 + h^2)}{b^2} \cdot \frac{WR \times 2\pi Rn \times R}{bh^3 \cdot C}$$

i.e. $$\delta = 7\pi \times \frac{WR^3 n}{C} \left[\frac{b^2 + h^2}{b^3 h^3}\right] \quad \text{...(14·12)}$$

By using more accurate value of α and β from Article 13·17 more accurate values of τ and δ can be calculated.

CIRCULAR-SECTION WIRE SPRINGS

Example 14·1. *A closely coiled helical spring is to carry a load of 500 N. Its mean coil diameter is to be 10 times that of the wire diameter. Calculate these diameters if the maximum shear stress in the material of the spring is to be 80 MN/m².*

Solution. Load to be carried, $W = 500$ N

Mean coil diameter, $D = 10\, d$ (wire diameter),

Shear stress, $\tau = 80 \text{ MN/m}^2$

$\frac{129}{1000}$
.12

Diameters, D and d:

Using the relation: $\tau = \frac{16\,WR}{\pi d^3}$, we have

$$80 \times 10^6 = \frac{16 \times 500 \times 5d}{\pi d^3}$$

$\therefore$ $$d^2 = \frac{16 \times 500 \times 5}{80 \times 10^6} = 1.591 \times 10^{-4}\ \text{m}^2$$

$\therefore$ $d = 0.0126$ m or **12·6 mm** **(Ans.)**

and, $D = 10\,d = 10 \times 12.6 = 126$ mm

i.e. $D =$ **126 mm** **(Ans.)**

Example 14·2. *A helical spring is made of 12 mm diameter steel wire wound on a 120 mm diameter mandrel. If there are 10 active coils, what is spring constant ? Take: C = 82 GN/m². What force must be applied to the spring to elongate it by 40 mm?*

Picture shows valve guides and spring mechanisms in an aircraft engine.

Solution. Diameter of steel wire,

$d = 12$ mm $= 0.012$ m

Diameter of mandrel, $D = 120$ mm $= 0.12$ m

Number of active coils,

$n = 10$

Modulus of rigidity, $C = 82$ GN/m²

Elongation of the spring,

$\delta = 40$ mm $= 0.04$ m

Spring constant:

We know that,

Spring constant = stiffness of spring (k),

$$k = \frac{W}{\delta} = \frac{Cd^4}{64R^3 n} = \frac{82 \times 10^9 \times (0.012)^4}{64 \times \left[\frac{0.12}{2}\right]^3 \times 10}$$

or, $k =$ **12300 N/m** **(Ans.)**

Force to be applied to the spring, W:

Again, $$\frac{W}{\delta} = 12300 \text{ or } \frac{W}{0.04} = 12300$$

or, $W =$ **492 N** **(Ans.)**

Example 14·3. (*a*) *Draw neat illustrative sketches to bring about the difference between a helical coil tension spring and helical coil compression spring.*

(*b*) *A helical coil spring is made of round steel wire 6·35 mm in diameter. The mean radius of helix is 31·75 mm, number of complete turns, 12; the spring is close-coiled. If C = 84·36 GN/m², find:*

(*i*) *The pull required to extend the spring by 25·4 mm, and*

(*ii*) *The stress in the wire.*

Solution. (*a*) The helical coil springs consist of a rod or wire wound in the form of helix. Fig. 14·2 (*a, b*) clearly indicates the difference between the helical coil tension and compression springs.

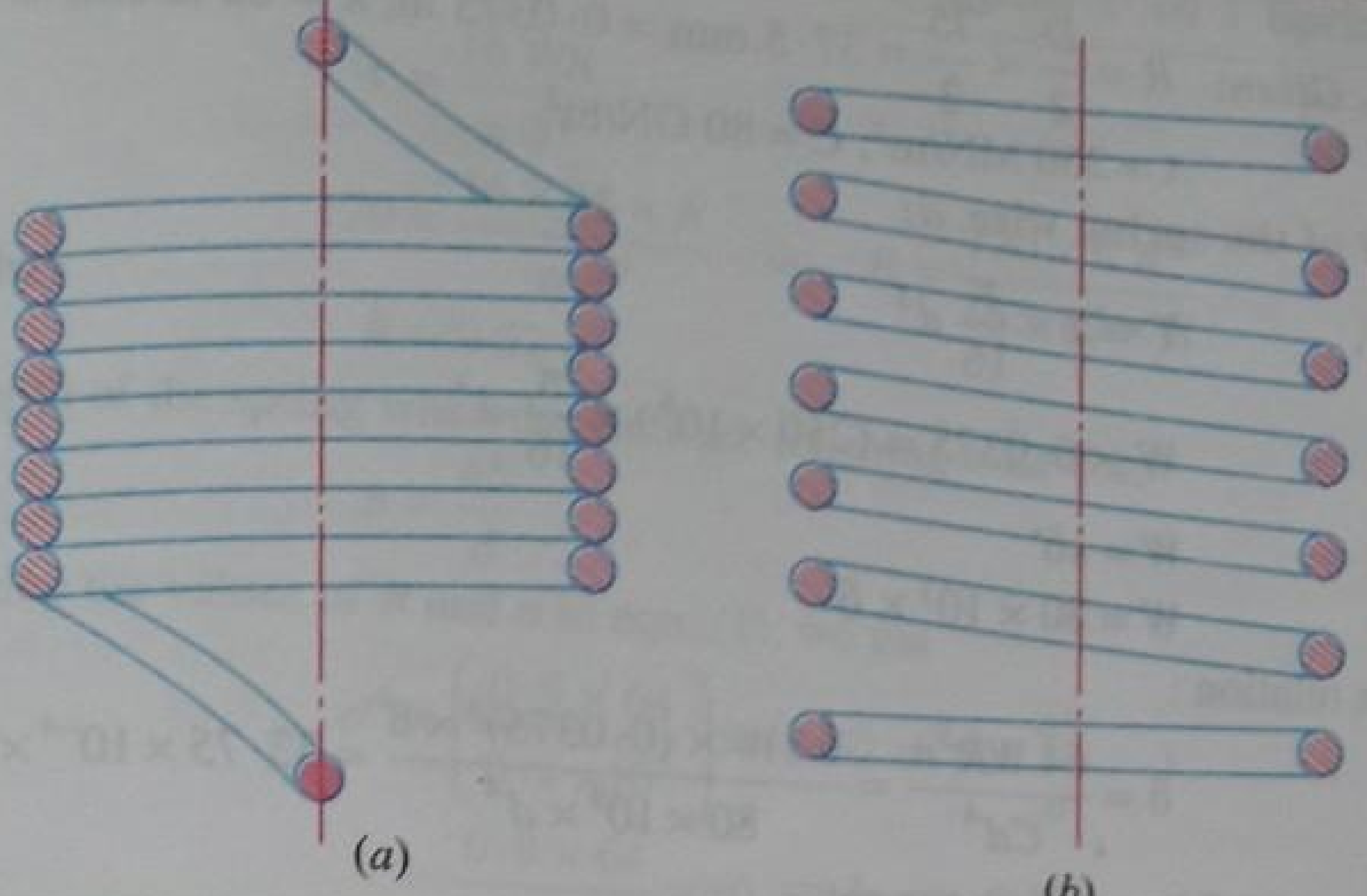


Fig. 14.2

It may be noted that in case of a helical tension spring it is not imperative to provide spacing between coils because helix angle is small while in case of helical compression spring, the helix angle being comparatively more, spacing is provided between the coils.

(b) Radius of the coil, $R = 31{\cdot}75$ mm $= 0{\cdot}03175$ m

Diameter of the wire, $d = 6{\cdot}35$ mm $= 0{\cdot}00635$ m

Number of coils, $n = 12$

Deflection of the spring, $= 25{\cdot}4$ mm $= 0{\cdot}0254$ m

(i) Pull required to extend the spring, *W*:

Using the relation,

$$\delta = \frac{64\,WR^3 n}{Cd^4}, \text{ we have}$$

$$0{\cdot}0254 = \frac{64W \times (0{\cdot}03175)^3 \times 12}{84{\cdot}36 \times 10^9 \times (0{\cdot}00635)^4}$$

$$W = \frac{0{\cdot}0254 \times 84{\cdot}36 \times 10^9 \times (0{\cdot}00635)^4}{64 \times (0{\cdot}03175)^3 \times 12}$$

or, $\quad W = 141{\cdot}7$ **N (Ans.)**

(ii) Shear stress in the wire, τ :

Using the relation,

$$\tau = \frac{16\,WR}{\pi d^3}, \text{ we have}$$

$$\tau = \frac{16 \times 141{\cdot}7 \times 0{\cdot}03175}{\pi \times (0{\cdot}00635)^3}$$

$$= 89{\cdot}5 \times 10^6 \text{ N/m}^2 = \mathbf{89{\cdot}5 \text{ MN/m}^2} \quad \textbf{(Ans.)}$$

Example 14·4. *A close-coiled helical spring has mean diameter of 75 mm and spring constant of 80 kN/m. It has 8 coils. What is the suitable diameter of the spring wire if maximum shear stress is not to exceed 250 MN/m² ? Modulus of rigidity of the spring wire material is 80 GN/m². What is the maximum axial load the spring can carry?* **(AMIE Summer, 2000)**

Solution. *Given*: $R = \frac{D}{2} = \frac{75}{2} = 37 \cdot 5 \text{ mm} = 0 \cdot 0375 \text{ m}; k = 80 \text{ kN/m}; n = 8$

$\tau = 250 \text{ MN/m}^2; C = 80 \text{ GN/m}^2.$

Diameter of the spring wire, *d*:

We know, $$T = \tau \times \frac{\pi}{16} d^3 \qquad \text{(where, } T = W \times R)$$

$$W \times 0 \cdot 0375 = (250 \times 10^6) \times \frac{\pi}{16} d^3 \qquad \text{...(i)}$$

Also, $$W = k\delta$$

or, $$W = 80 \times 10^3 \times \delta \qquad \text{...(ii)}$$

Using the relation :

$$\delta = \frac{64 WR^3 n}{Cd^4} = \frac{64 W \times (0 \cdot 0375)^3 \times 8}{80 \times 10^9 \times d^4} = 33 \cdot 75 \times 10^{-4} \times \frac{W}{d^4}$$

Substituting for δ in eqn. (*ii*), we get

$$W = 80 \times 10^3 \times 33{\cdot}75 \times 10^{-14} \times \frac{W}{d^4}$$

or, $$d^4 = 80 \times 10^3 \times 33{\cdot}75 \times 10^{-14}$$

$\therefore$ $$d = 0{\cdot}0128 \text{ m or } 12{\cdot}8 \text{ mm} \quad \textbf{(Ans.)}$$

Maximum axial load the spring can carry *W* :

From eqn. (*i*), we have

$$W \times 0 \cdot 0375 = (250 \times 10^6) \times \frac{\pi}{16} \times (0 \cdot 0128)^3$$

$\therefore$ $$W = 2745{\cdot}2 \text{ N} \quad \textbf{(Ans.)}$$

Example 14·5. *A close-coiled helical spring is to have a stiffness of 900 N/m in compression, with a maximum load of 45N and a maximum shearing stress of 120 N/mm². The solid length of the spring (i.e., coils touching) is 45 mm. Find:*

(*i*) *The wire diameter,*

(*ii*) *The mean coil radius, and*

(*iii*) *The number of coils.*

Take modulus of rigidity of material of the spring $= 0{\cdot}4 \times 10^5$ *N/mm²*.

Solution. *Given*: $k = 900 \text{ N/m} = 0{\cdot}9 \text{ N/mm}$; $W = 45 \text{ N}; \tau = 120 \text{ N/mm}^2; C = 0{\cdot}4 \times 10^5 \text{ N/mm}^2$

(*i*) **The wire diameter, *d*:**

$$\delta = \frac{64 WR^3 n}{Cd^4}$$

or, $$k = \frac{W}{\delta} = \frac{Cd^4}{64 R^3 n}$$

or, $$0 \cdot 9 = \frac{0 \cdot 4 \times 10^5 \times d^4}{64 R^3 n}$$

or, $$d^4 = \left(\frac{0 \cdot 9 \times 64}{0 \cdot 4 \times 10^5} \right) R^3 n \qquad \text{...(1)}$$

Triggering mechanisms of guns are operated by springs besides other components.

Also, $\tau = \dfrac{16\ WR}{\pi\ d^3}$

$\therefore$ $120 = \dfrac{16 \times 45 \times R}{\pi\ d^3}$ or $R = \dfrac{120 \times \pi\ d^3}{16 \times 45 \times R}$

or, $R = 0{\cdot}52\ d^3$ ...(2)

Solid length of the spring when the coils are touching $= nd = 45$

$\therefore$ $n = \dfrac{45}{d}$ ...(3)

Substituting the values of R and n in eqn. (1), we get

$$d^4 = \left(\frac{0{\cdot}9 \times 64}{0{\cdot}4 \times 10^5}\right) \times (0{\cdot}52\ d^3)^3 \times \frac{45}{d}$$

$$= \frac{0{\cdot}9 \times 64}{0{\cdot}4 \times 10^5} \times (0{\cdot}52)^3 \times 45 \times d^8$$

or, $$d^4 = \frac{0{\cdot}4 \times 10^5}{0{\cdot}9 \times 64 \times (0{\cdot}52)^3 \times 45} = 109{\cdot}75$$

$\therefore$ $d = (109{\cdot}75)^{1/4} \simeq \mathbf{3{\cdot}24\ mm}$ **(Ans.)**

***(ii)* The mean coil radius, *R*:**

$R = 0{\cdot}52\ d^3$ ... [Eqn. (2)]

$= 0{\cdot}52 \times (3{\cdot}24)^3 = \mathbf{17{\cdot}68\ mm}$ **(Ans.)**

***(iii)* The number of coils, *n*:**

$n = \dfrac{45}{d}$ ... [Eqn. (3)]

$= \dfrac{45}{3{\cdot}24} = \mathbf{13{\cdot}88}$ **(Ans.)**

Example 14·6. *A close-coiled helical spring of 100 mm mean diameter is made of 10 mm diameter rod and has 20 turns. The spring carries an axial load of 200 N. Determine the shearing stress. Taking the value of modulus of rigidity = 84 GN/m² determine the deflection when carrying this load. Also calculate the stiffness of the spring and frequency of free vibrations for a mass hanging from it.*

Solution. Mean diameter of the spring,

$D = 100\ \text{mm} = 0{\cdot}1\ \text{m}$

or, $R = 0{\cdot}1/2 = 0{\cdot}05\ \text{m}$

Diameter of the rod, $d = 10\ \text{mm} = 0{\cdot}01\ \text{m}$

Axial load, $W = 200\ \text{N}$;

No. of turns, $n = 20$

$C = 84\ \text{GN/m}^2$.

Shear stress, τ :

Using the relation: $\tau = \dfrac{16\ WR}{\pi d^3}$, we have

$$\tau = \frac{16 \times 200 \times 0{\cdot}05}{\pi \times (0{\cdot}01)^3}$$

$= 50{\cdot}93 \times 10^6\ \text{N/m}^2$ **50·93 MN/m²** **(Ans.)**

Deflection of the spring, δ :

Using the relation: $\delta = \dfrac{64\ WR^3 n}{Cd^4}$, we have $\delta = \dfrac{64 \times 200 \times (0{\cdot}05)^3 \times 20}{84 \times 10^9 \times (0{\cdot}01)^4}$

or, $\delta = 0{\cdot}03809$ m or **38·09 mm** **(Ans.)**

Stiffness, k:

$$k = \frac{W}{\delta} = \frac{200}{38 \cdot 09} = 5 \cdot 25 \text{ N/mm} \quad \textbf{(Ans.)}$$

Frequency of free vibration, f:

Using the relation: $f = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$, we get $f = \frac{1}{2\pi}\sqrt{\frac{9 \cdot 81}{0 \cdot 03809}}$

$= 2{\cdot}55$ **vibrations/s** **(Ans.)**

Example 14·7. *A close - coiled helical spring is made out of 10 mm diameter steel rod. The coil consists of 10 complete turns with a mean diameter of 120 mm. The spring carries an axial pull of 200 N. Find the maximum shear stress induced in the section of the rod. If $C = 80$ GN/m^2, find the deflection in the spring, the stiffness and strain energy stored in the spring.*

Solution. Diameter of steel rod,

$d = 10$ mm or $0{\cdot}01$ m

Number of turns, $n = 10$

Mean radius of each turn, $R = \frac{120}{2} = 60$ mm or $0 \cdot 06$ m

Axial pull, $W = 200$ N; $C = 80$ GN/m^2

Deflection of the spring, δ :

Using the relation: $\delta = \frac{64\, WR^3 n}{Cd^4}$, we have $\delta = \frac{64 \times 200 \times (0 \cdot 06)^3 \times 10}{80 \times 10^9 \times (0 \cdot 01)^4}$

$= 0{\cdot}03456$ m or $=$ **34·56 mm** **(Ans.)**

Stiffness of the spring, k:

Using the relation: $k = \frac{W}{\delta}$, we have $k = \frac{200}{34 \cdot 56} = 5 \cdot 79$ **N/mm** **(Ans.)**

Maximum shear stress, τ :

We know, $\tau = \frac{16\, WR}{\pi\, d^3}$ or $\tau = \frac{16 \times 200 \times 0 \cdot 06}{\pi \times (0 \cdot 01)^3}$

$= 61{\cdot}11 \times 10^6$ N/m^2 $=$ **61·11 MN/m^2** **(Ans.)**

Strain energy stored, U:

Using the relation: $U = \frac{1}{2} \cdot W \cdot \delta$, we have $U = \frac{1}{2} \times 200 \times 0 \cdot 03456$

$=$**3·456 Nm** **(Ans.)**

Example 14·8. *For a close-coiled helical spring subjected to an axial load of 300 N having 12 coils of wire diameter of 16 mm, and made with coil diameter of 250 mm, find:*

(i) Axial deflection;
(ii) Strain energy stored;
(iii) Maximum torsional shear stress in the wire;
(iv) Maximum shear stress using Wahl's correction factor.

Take: $C = 80$ GN/m^2

Solution. Number of coils, $n = 12$ coils

Wire diameter, $d = 16$ mm $= 0{\cdot}016$ m

Coil diameter, $D = 250$ mm $= 0{\cdot}25$ m

Modulus of rigidity, $C = 80$ GN/m^2

Axial load, $W = 300$ N

(i) **Axial deflection, δ :**

$$\delta = \frac{64\,WR^3 n}{Cd^4} = \frac{64 \times 300 \times (0 \cdot 25/2)^3 \times 12}{80 \times 10^9 \times (0 \cdot 016)^4} \text{ m}$$

$= 0{\cdot}0858$ m or **85·8 mm** (Ans.)

(ii) **Strain energy stored, U:**

$$U = \frac{1}{2} W\delta = \frac{1}{2} \times 300 \times 0 \cdot 0858 = \mathbf{12{\cdot}87\ Nm} \quad \text{(Ans.)}$$

(iii) **Maximum torsional shear stress, τ :**

$$\tau = \frac{16\,WR}{\pi\,d^3} = \frac{16 \times 300 \times (0 \cdot 25/2)}{\pi \times (0 \cdot 016)^3} \times 10^{-6}\ \text{MN/m}^2$$

$= 46.63$ MN/m^2

i.e. $\tau = \mathbf{46{\cdot}63\ MN/m^2}$ (Ans.)

(iv) **Maximum shear stress using Wahl's correction factor, τ :**

$$\tau = \frac{16\,WR}{\pi\,d^3} \times K, \quad \text{where } K = \frac{4S - 1}{4S - 4} + \frac{0 \cdot 615}{S}$$

But S (spring index) $= \dfrac{D}{d} = \dfrac{250}{16} = 15 \cdot 625$

$$\therefore \quad K = \frac{4 \times 15 \cdot 625 - 1}{4 \times 15 \cdot 625 - 4} + \frac{0 \cdot 615}{15 \cdot 625} = 1 \cdot 0513 + 0 \cdot 0394 = 1 \cdot 0907$$

$$\therefore \quad \tau = \frac{16 \times 300 \times (0 \cdot 25/2)}{\pi \times (0 \cdot 016)^3} \times 1 \cdot 0907 \times 10^{-6}\ MN/\text{m}^2$$

$= 50{\cdot}85$ MN/m^2 *i.e.* $\tau = \mathbf{50{\cdot}85\ MN/m^2}$ (Ans.)

Example 14·9. *A safety valve of 76 mm diameter is to blow off at a pressure of 1·12 MN/m². It is held by a close-coiled compression spring of circular steel bar. The mean diameter is 152·5 mm and the initial compression is 25·4 mm. Find the diameter of the steel bar and the number of turns necessary if the stress allowed is 126 MN/m² and C = 79 GN/m².*

Solution. Force required to lift the valve, $= p \times \pi/4 \times d_v^2$

[where, p = intensity of pressure; d_v = dia. of the valve.]

$= 1{\cdot}12 \times 10^6 \times \pi/4\,(76/1000)^2 = 5080$ N

Shock absorbers.

We know that,

$$\tau = \frac{16\,WR}{\pi\,d^3}$$

$$\therefore \quad 126 \times 10^6 = \frac{16 \times 5080 \times \left[\dfrac{152 \cdot 5}{2 \times 1000}\right]}{\pi \times d^3}$$

or,
$$d^3 = \frac{16 \times 5080 \times \left[\dfrac{152 \cdot 5}{2 \times 1000}\right]}{126 \times 10^6 \times \pi}$$

$\therefore \quad d = 0{\cdot}025$ m or **25 mm** (Ans.)

Also, $$\delta = \frac{64\,WR^3 n}{Cd^4}$$

$$\frac{25\cdot4}{1000} = \frac{64\times5080\times\left[\frac{152\cdot5}{2\times1000}\right]^3\times n}{79\times10^9\times\left(\frac{25}{1000}\right)^4}$$

$$0\cdot0254 = \frac{144\cdot13\,n}{30859.4}$$

i.e. $$n = \frac{0\cdot0254\times30859.4}{144\cdot13} = 5\cdot4 \quad \textbf{(Ans.)}$$

Shock absorbers.

Example 14·10. *A railway wagon weighing 40 kN and moving with a speed of 8 km / h is stopped by a buffer of 4 springs whose allowable maximum compression is 150 mm. Find out the number of turns in each spring, if the diameter of the spring wire is 14 mm and the diameter of coil is 80 mm. Assume $C = 84$ GN/m².*

Solution. Weight of railway wagon,

$$W = 40 \text{ kN}$$

Speed of the wagon, $$v = 8 \text{ km / h}$$

$$= \frac{8\times1000}{60\times60} = 2\cdot22 \text{ m/s}$$

K.E. of the wagon $$= \frac{1}{2}\cdot\frac{W\,v^2}{g} = \frac{1}{2}\times\frac{40\times1000}{9\cdot81}\times2\cdot22^2 = 10048 \text{ Nm}$$

Energy absorbed by each spring of the buffer

$$= \frac{10048}{4} = 2512 \text{ Nm}$$

Also, energy absorbed = Work done

$$2512 = \frac{1}{2}W\delta = \frac{1}{2}\times W\times(150/1000)$$

$$W = \frac{2512\times2\times1000}{150} = 33493 \text{ N}$$

Number of coils, *n*:

Using the relation: $$\delta = \frac{64\,WR^3\,n}{Cd^4}, \text{ we have}$$

$$0 \cdot 15 = \frac{64 \times 33493 \times (0 \cdot 04)^3 \times n}{84 \times 10^9 \times (0 \cdot 014)^4}$$

$$\therefore \quad n = \frac{0 \cdot 15 \times 84 \times 10^9 \times (0 \cdot 014)^4}{64 \times 33493 \times (0 \cdot 04)^3} = 3 \cdot 53 \quad \textbf{(Ans.)}$$

Example 14·11. *A weight of 200 N is dropped on to a helical spring made of 15 mm wire closely coiled to a mean diameter of 120 mm with 20 coils. Determine the height of drop if the instantaneous compression is 80 mm.*

Assume: $C = 84\ GN/m^2$.

Solution. Magnitude of falling weight, $W = 200\ N$

Diameter of wire, $d = 15$ mm or $0{\cdot}015$ m

Mean diameter of coils, $D = 120$ mm or $0{\cdot}12$ m

Number of coils, $n = 20$.

Instantaneous compression, $\delta = 80$ mm or $0{\cdot}08$ m

Height of drop, *h*:

Using the relation: $\delta = \dfrac{64\ WR^3\ n}{Cd^4}$, we have

$$0 \cdot 08 = \frac{64\ W \times (0 \cdot 06)^3 \times 20}{84 \times 10^9 \times (0 \cdot 015)^4}$$

$$W = \frac{0 \cdot 08 \times 84 \times 10^9 \times (0 \cdot 015)^4}{64 \times (0 \cdot 06)^3 \times 20} = 1230 \text{ N}$$

(where, W = gradually applied load)

Also, energy supplied by the impact load = Energy stored

$$P\ (h + \delta) = \frac{1}{2}\ W \delta$$

$$200\ (h + 0 \cdot 08) = \frac{1}{2} \times 1230 \times 0 \cdot 08$$

$$h + 0{\cdot}08 = 0{\cdot}246$$

$h = 0{\cdot}166$ m or **166 mm** **(Ans.)**

Example 14·12. *Determine amount of compression and maximum shear stress produced when a load of 2100 N is dropped axially on a close-coiled helical spring from a height of 240 mm. The spring has 22 coils each of mean diameter 180 mm and wire diameter is 25 mm, C = 84000 N/mm².*

Solution. Diameter of wire, $d = 25$ mm

Mean diameter of coil, $D = 180$ mm

Number of coils, $n = 22$

Height of fall, $h = 240$ mm

Falling load, $P = 2100$ N

Modulus of rigidity, $C = 84000\ \text{N/mm}^2$.

Amount of compression, δ (mm) :

Let, W_e = Equivalent gradually applied load which shall produce the same effect as produced by the given falling load of 2100 N

Now, work done by the falling load = Work done by W_e

$$P\,(h+\delta) = \frac{1}{2}\cdot W_e\cdot\delta \qquad \left[\because\ \delta = \frac{64\,W_e R^3 n}{Cd^4}\right.$$

$$2100(240+\delta) = \frac{1}{2}\,\frac{\delta\,Cd^4}{64\,R^3 n}\cdot\delta \qquad \left.\therefore\ W_e = \frac{\delta\,Cd^4}{64\,R^3 n}\right]$$

or,
$$2100\,(240+\delta) = \frac{1}{2}\cdot\frac{84000\times(25)^4}{64\times(180/2)^3\times 22}\cdot\delta^2 = 15\cdot 98\,\delta^2$$

or,
$$\delta^2 - 131{\cdot}41\delta - 31539 = 0$$

or,
$$\delta = \frac{131\cdot 41 \pm \sqrt{131\cdot 41^2 + 4\times 31539}}{2} = 255 \text{ mm}$$

i.e.
$$\delta = 255 \text{ mm} \quad \textbf{(Ans.)}$$

Maximum shear stress, τ :

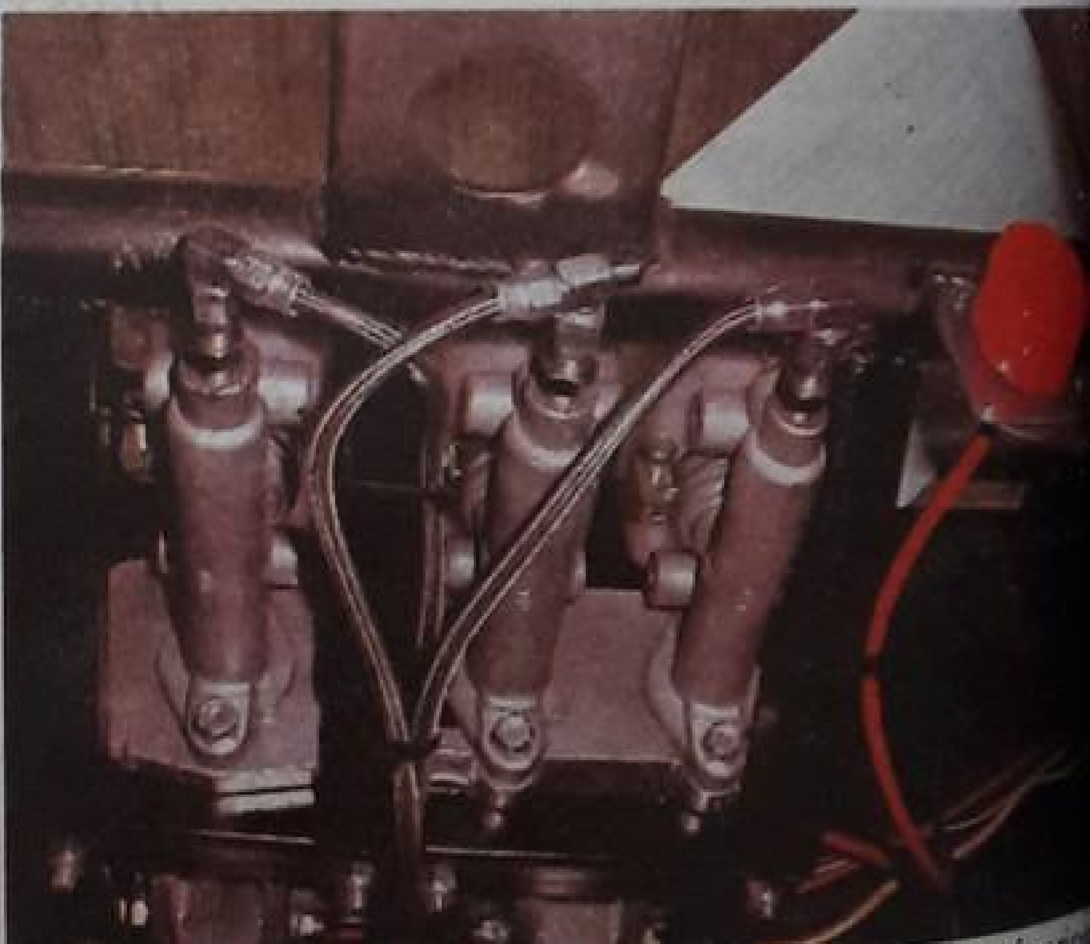
Now substituting the value of δ in the following relation, we get

$$W_e = \frac{\delta\,Cd^4}{64\,R^3 n} = \frac{255\times 84000\times(25)^4}{64\times 90^3\times 22} = 8151 \text{ N}$$

$$\therefore \quad \tau = \frac{16\,W_e R}{\pi\,d^3} = \frac{16\times 8151\times 90}{\pi\times(25)^3} = 239 \text{ N/mm}^2$$

Hence,
$$\tau = 239 \text{ N/mm}^2 \quad \textbf{(Ans.)}$$

14·3·2. Subjected to 'Axial twist'



Braking mechanisms of automobiles consist of springs and master cylinders (as shown above).

When a twisting couple is applied to the spring parallel to the axis of the spring wire it produces a bending effect on it. Depending upon the direction of application of the twisting couple or turning moment the spring coils will close or open out. In both cases the radius of coils will close or open out. In both cases the radius of coils changes and bending stresses will be induced.

Let, n_1 = Number of coils before application of twisting moment,

n_2 = Number of coils after application of twisting moment,

ϕ = Angle of rotation,

I = Moment of inertia of coil section,

R_1 = Mean radius of coil,

R_2 = Changed radius of coil,

σ_b = Bending stress, and

E = Young's modulus of elasticity.

Initial curvature $= \dfrac{1}{R_1}$

Final curvature $= \dfrac{1}{R_2}$

Change in curvature $= \frac{1}{R_2} - \frac{1}{R_1}$

Also, as per bending equation,

$$\frac{M}{I} = \frac{E}{R} \text{ or } \frac{1}{R} = \frac{M}{EI}$$

$$\frac{M}{EI} = \frac{1}{R_2} - \frac{1}{R_1}$$

Since length of the wire remains unchanged before and after applying the twisting couple,

$\therefore$ $$l = 2\pi R_1 n_1 = 2\pi R_2 n_2$$

But, ϕ = Final helix angle – initial helix angle

$$= (2\pi n_2 - 2\pi n_1)$$

$$\frac{M}{EI} = \frac{1}{R_2} - \frac{1}{R_1} = \frac{1}{\frac{l}{2\pi n_2}} - \frac{1}{\frac{l}{2\pi n_1}} = \frac{2\pi}{l}(n_2 - n_1) = \frac{\phi}{l} \quad ...(14\cdot13)$$

$\therefore$ $$\phi = \frac{Ml}{EI} \quad ...(14\cdot14)$$

$$= \frac{M \times 2\pi Rn}{E \times \frac{\pi}{64} d^4} = \frac{128\, MRn}{Ed^4}$$

Also, $$\sigma_b = \frac{M}{Z} = \frac{My}{I} = \frac{Md/2}{\frac{\pi d^4}{64}} = \frac{32\, M}{\pi\, d^3} \quad ...(14\cdot15)$$

Now, energy stored, $$U = \frac{1}{2} M\,\phi = \frac{1}{2} M \cdot \frac{Ml}{EI} = \frac{1}{2}\frac{M^2 l}{EI}$$

$$= \frac{1}{2} \cdot \frac{\sigma_b \cdot \pi\, d^3}{32} \times \frac{\sigma_b \cdot \pi d^3}{32} \times \frac{l \times 64}{E \times \pi d^4}$$

$$= \frac{\pi^2 d^6 \sigma_b^2 l \times 64}{2 \times 32 \times 32 \times 6 \times E \times \pi d^4} = \frac{\sigma_b^2 \times \pi d^2 l}{4 \times 8E} = \frac{\sigma_b^2}{8E} \times \frac{\pi d^2 l}{4}$$

$$= \frac{\sigma_b^2}{8E} \times \text{volume of spring wire}$$

Hence, energy stored, $$U = \frac{\sigma_b^2}{8E} \times \text{volume of spring} \quad ...(14\cdot16)$$

Example 14·13. *A closely coiled helical spring made of wire 5 mm in diameter and having an inside diameter of 40 mm joins two shafts. The effective number of coils between the shafts is 15 and 0·735 kW is transmitted through the spring at 1000 r.p.m. Calculate the relative axial twist in degrees between the ends of spring and also the intensity of bearing stress in the material. E = 200 GN/m².*

Solution. $$d = 5 \text{ mm} = 0{\cdot}005 \text{ m}$$

Mean diameter of the coil, $D = 0{\cdot}04 + 0{\cdot}005 = 0{\cdot}045$ m

or, $$R = \frac{0{\cdot}045}{2} = 0{\cdot}0225 \text{ m}$$

We know that, $$P = \frac{2\pi NT}{60 \times 1000}$$

or, $$0.735 = \frac{2\pi MN}{60 \times 1000} \quad (\because M = T)$$

or, $$M = \frac{0.735 \times 60 \times 1000}{2\pi \times 1000} = 7 \text{ Nm}$$

Now, $$\phi = \frac{128\, MRn}{Ed^4} = \frac{128 \times 7 \times 0.0225 \times 15}{200 \times 10^9 \times (0.005)^4}$$

or, $$\phi = 2.4 \text{ radians} = \mathbf{137.5°} \quad \textbf{(Ans.)}$$

$$\sigma_b = \frac{32M}{\pi d^3} = \frac{32 \times 7}{\pi\,(0.005)^3} = \mathbf{570.4\ MN/m^2} \quad \textbf{(Ans.)}$$

Example 14·14. *A close-coiled helical spring made of round steel wire 6 mm diameter, having 10 complete turns is subjected to an axial couple M. The mean coil radius is 42 mm. If the maximum bending stress in spring wire is not to exceed 240 MN/m², determine:*

(i) The magnitude of axial couple M;

(ii) The angle through which one end of spring is turned relative to the other end.

Take: $E_{steel} = 200$ *GN/m².*

Solution. Diameter of steel wire,

$$d = 6 \text{ mm} = 0.006 \text{ m}$$

Number of complete turns, $n = 10$

Mean coil radius, $R = 42 \text{ mm} = 0.042 \text{ m}$

Maximum bending stress, $\sigma_b = 240 \text{ MN/m}^2$

$E_{steel} = 200 \text{ GN/m}^2$.

(i) Axial couple, M:

Using the relation: $\sigma_b = \dfrac{32M}{\pi d^3}$, we have $240 \times 10^6 = \dfrac{32M}{\pi \times (0.006)^3}$

$$\therefore \quad M = \frac{240 \times 10^6 \times \pi \times (0.006)^3}{32} \text{ Nm} = \mathbf{5.089\ Nm} \quad \textbf{(Ans.)}$$

(ii) Angle of rotation ϕ:

Using the relation: $\phi = \dfrac{128\, MRn}{Ed^4}$, we have

$$\phi = \frac{128 \times 5.089 \times 0.042 \times 10}{200 \times 10^9 \times (0.006)^4} \times \frac{180}{\pi} \text{ degrees} = 60.47°$$

Hence, $\phi = \mathbf{60.47°}$ **(Ans.)**

14·4. OPEN-COILED HELICAL SPRINGS

14·4·1. With Axial Load

Employing the same symbols as used in previous articles, the *slope of coils* α *is introduced additionally.*

The wire length is now $l = 2\pi R \sec\alpha \times n$, where R is the radius of coil.

The couple applied to the material under the applied load W will be WR, and at each point along the centre line of wire this couple may be resolved into two components, one of **torsion** and one of **bending.**

The couple producing *torsion,* $T = WR \cos\alpha$

The couple producing *bending,* $M = WR \sin\alpha$.

Refer to Fig. 14·3. The couple WR will act in a plane passing through the axes OY and OX, the centre line of the wire being at angle α to OX.

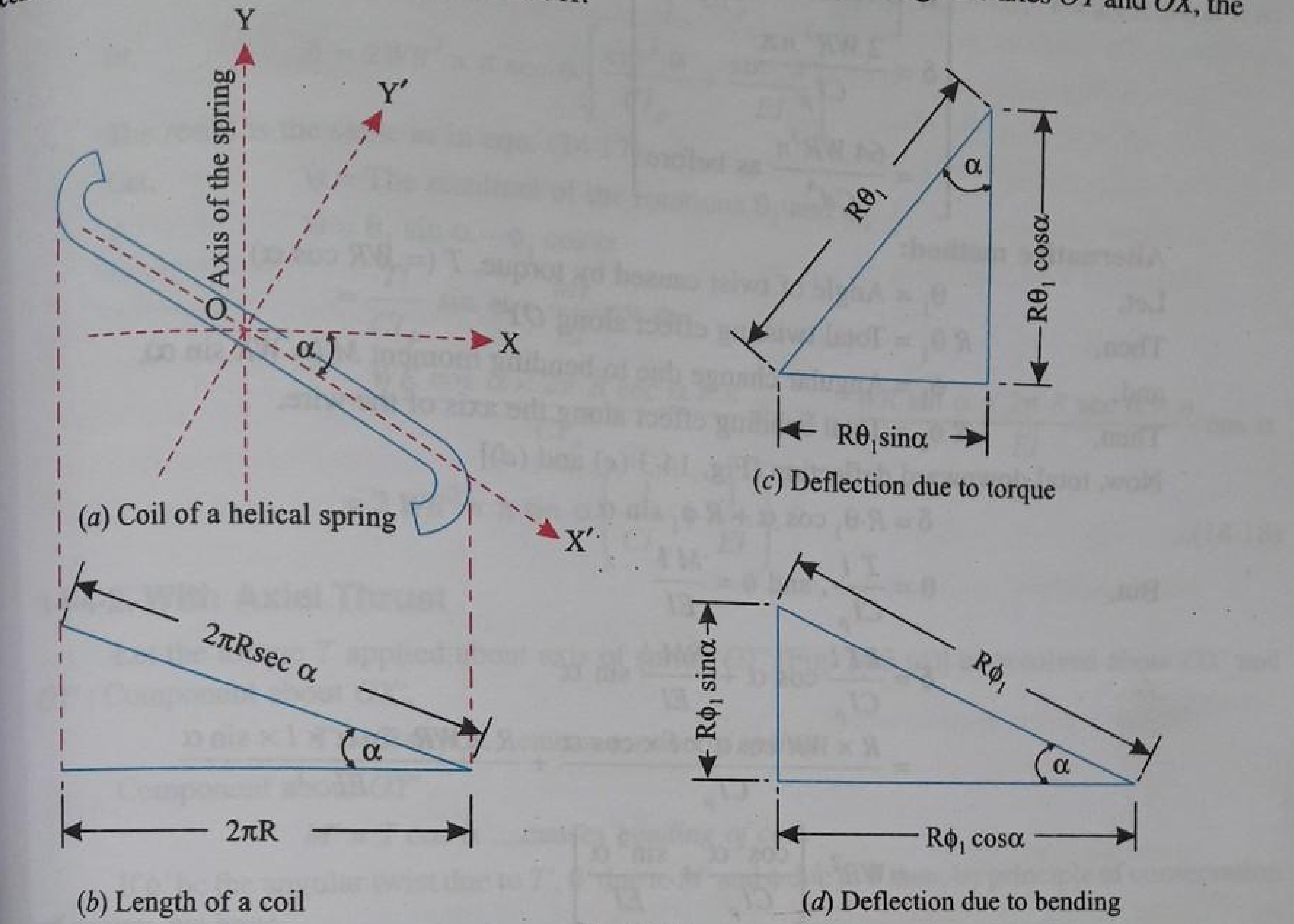


Fig. 14.3

The bending moment will tend to wind the coils of the spring more tightly, and to a smaller radius of curvature.

The axial extension of the spring may be most easily calculated by *equating the work done by the load to the internal strain energy of the material.*

The *strain energy due to bending* $= \dfrac{M^2 l}{2EI}$

The strain energy for twisting $= \dfrac{T^2 l}{2\,CI_p}$

∴ Where the deflection is δ,

$$\frac{1}{2} W \cdot \delta = \frac{T^2 l}{2\,CI_p} + \frac{M^2 l}{2\,EI} = \frac{W^2 R^2 \cos^2\alpha \cdot l}{2\,CI_p} + \frac{W^2 R^2 \sin^2\alpha\, l}{2\,EI}$$

$$= \frac{W^2 R^2 l}{2}\left[\frac{\cos^2\alpha}{CI_p} + \frac{\sin^2\alpha}{EI}\right]$$

$$= \frac{W^2 R^2\, 2\pi\, Rn \sec\alpha}{2}\left[\frac{\cos^2\alpha}{CI_p} + \frac{\sin^2\alpha}{EI}\right]$$

∴ Deflection,

$$\delta = 2\,WR^3\, n\, \pi\, \sec\alpha \left[\frac{\cos^2\alpha}{CI_p} + \frac{\sin^2\alpha}{EI}\right] \qquad ...(14\cdot17)$$

$$\left[\begin{aligned} &\text{If } \alpha \text{ is taken as zero,} \\ &\delta = \frac{2\,WR^3\,n\pi}{CI_p} \\ &\;\;= \frac{64\,WR^3 n}{Cd^4} \text{ as before} \end{aligned}\right]$$

Alternative method:

Let, θ_1 = Angle of twist caused by torque, T (= $WR \cos \alpha$)

Then, $R\,\theta_1$ = Total twisting effect along OY'

and, ϕ_1 = Angular change due to bending moment M (= $WR \sin \alpha$),

Then, $R\,\phi_1$ = Total bending effect along the axis of the wire.

Now, total downward deflection [Fig. 14·3 (*c*) and (*d*)]

$$\delta = R\,\theta_1 \cos \alpha + R\,\phi_1 \sin \alpha$$

But,
$$\theta = \frac{T\,l}{CI_p}, \text{ and } \phi = \frac{M\,l}{EI}$$

$\therefore$
$$\delta = \frac{RT\,l}{CI_p} \cos \alpha + \frac{RM\,l}{EI} \sin \alpha$$

$$= \frac{R \times WR \cos \alpha \times l \times \cos \alpha}{CI_p} + \frac{R \times WR \sin \alpha \times l \times \sin \alpha}{EI}$$

$$= WR^2\,l \left[\frac{\cos^2 \alpha}{CI_p} + \frac{\sin^2 \alpha}{EI}\right]$$

Shock-absorber of a motorcycle.

$$= WR^2 \times 2\pi\, R \sec\alpha \times n \left[\frac{\cos^2\alpha}{CI_p} + \frac{\sin^2\alpha}{EI}\right] \qquad (\because\ l = 2\pi\, R \sec\alpha \times n)$$

or,
$$\delta = 2WR^3\, n\, \pi \sec\alpha \left[\frac{\cos^2\alpha}{CI_p} + \frac{\sin^2\alpha}{EI}\right]$$

The result is the same as in eqn. (14·17)

Let, ψ = The resultant of the rotations θ_1 and α_1.

$$\therefore \quad \psi = \theta_1 \sin\alpha - \phi_1 \cos\alpha$$

$$= \frac{Tl}{CI_p}\sin\alpha - \frac{Ml}{EI}\cos\alpha$$

$$= \frac{WR\cos\alpha \times 2\pi\, R\sec\alpha \times n}{CI_p}\sin\alpha - \frac{WR\sin\alpha \times 2\pi\, R\sec\alpha \times n}{EI}\cos\alpha$$

$$= 2\, WR^2\, n\, \pi \sin\alpha \left(\frac{1}{CI_p} - \frac{1}{EI}\right) \qquad \text{...(14·18)}$$

14·4·2. With Axial Thrust

Let the torque T applied about axis of spring OY' [Fig. 14·3 (*a*)] be resolved about OX' and OY'; Component about OX',

$T' = T\sin\alpha$ ....causes *torsion of spring*

Component about OY',

$M' = T\cos\alpha$ ....causes *bending of coil.*

If ϕ' be the angular twist due to T', θ' due to M' and ϕ due to T then, by principle of conservation of energy, we have

$$\frac{1}{2}T\phi = \frac{1}{2}T'\phi' + \frac{1}{2}M'\theta' = \frac{1}{2}T' \times \frac{T'\,l}{CI_p} + \frac{1}{2}M' \times \frac{M'\,l}{EI}$$

$$= \frac{1}{2}T'^2\frac{l}{CI_p} + \frac{1}{2}M'^2\frac{l}{EI} = \frac{1}{2}\frac{T^2\sin^2\alpha \cdot l}{CI_p} + \frac{1}{2}\frac{T^2\cos^2\alpha \times l}{EI}$$

$$\therefore \quad \phi = \frac{T\sin^2\alpha \cdot l}{CI_p} + \frac{T\cos^2\alpha \cdot l}{EI}$$

But, $l = 2\pi\, R\sec\alpha \times n$

$$\therefore \quad \phi = 2\, TR\, n\, \pi \sec\alpha \left[\frac{\sin^2\alpha}{CI_p} + \frac{\cos^2\alpha}{EI}\right] \qquad \text{...(14·19)}$$

For axial deflection/extension resolve rotations as before:

$$\delta = TRl\sin\alpha\cos\alpha\left(\frac{1}{CI_p} - \frac{1}{EI}\right)$$

$$= TR \times 2\pi\, Rn\sec\alpha \times \sin\alpha\cos\alpha\left(\frac{1}{CI_p} - \frac{1}{EI}\right)$$

$$= 2\, TR^2\, n\, \pi\sin\alpha\left(\frac{1}{CI_p} - \frac{1}{EI}\right) \qquad \text{...(14·20)}$$

14·4·3. Stresses in Circular Wire of Open Coil Spring

Consider on open coil helical spring of a mean coil radius R, angle of helix α and wire diameter d. The stresses in the circular wire are calculated as follows:

Case I. Subjected to an axial load W:

On any section of the spring wire,

Twisting moment, $T = WR \cos \alpha$

Bending moment, $M = WR \sin \alpha$

Maximum torsional stress on any section,

$$\tau_1 = \frac{16\,T}{\pi\,d^3} = \frac{16\,WR \cos \alpha}{\pi\,d^3} \qquad ...(i)$$

Direct shear stress due to axial load,

$$\tau_2 = \frac{W}{\frac{\pi}{4} \times d^2} = \frac{4\,W}{\pi\,d^2} \qquad ...(ii)$$

Maximum shear stress at inner coil radius,

$$\tau = \tau_1 + \tau_2 = \frac{16\,W \cos \alpha}{\pi\,d^3} + \frac{4\,W}{\pi\,d^2} \qquad ...(14{\cdot}21)$$

Minimum shear stress at outer coil radius,

$$\tau = \tau_1 - \tau_2 = \frac{16\,WR \cos \alpha}{\pi\,d^3} - \frac{4\,W}{\pi\,d^2} \qquad ...(14{\cdot}22)$$

Maximum stress due to bending,

$$\sigma_b = \frac{M}{Z} = \frac{WR \sin \alpha}{\frac{\pi\,d^3}{32}} = \frac{32\,WR \sin \alpha}{\pi\,d^3} \qquad ...(14{\cdot}23)$$

Maximum principal stress occurs at the inner coil radius.

Principal stresses: $$\sigma_{max}, \sigma_{min} = \frac{\sigma_b}{2} \pm \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2} \qquad ...(14{\cdot}24)$$

Case II. Subjected to axial torque T:

Twisting moment, $T' = T \sin \alpha$

Bending moment, $M' = T \cos \alpha$

Maximum torsional shear stress due to T',

$$\tau = \frac{16\,T'}{\pi\,d^3} = \frac{16\,T \sin \alpha}{\pi\,d^3} \qquad ...(i)$$

Maximum stress due to bending,

$$\sigma_b = \frac{32\,M'}{\pi\,d^3} = \frac{32\,T \cos \alpha}{\pi\,d^3} \qquad ...(ii)$$

Principal stresses at the extreme radii (inner and outer radii of coil),

$$\sigma_{max}, \sigma_{min} = \frac{\sigma_b}{2} \pm \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2} \qquad ...(14{\cdot}25)$$

Example 14·15. *An open-coiled helical spring made from wire of circular cross-section is required to carry a load of 120 N. The wire diameter is 8 mm and mean coil radius is 48 mm. If the helix angle of the spring is 30° and the number of turns is 12, calculate:*

(i) Axial deflection;

(ii) Angular rotation of free end with respect to the fixed end of the spring.

Take: $C_{steel} = 80\ GN/m^2;\ E_{steel} = 200\ GN/m^2.$

Solution. Diameter of wire, $d = 8\text{ mm} = 0.008\text{ m}$

Mean radius of coil, $R = 48\text{ mm} = 0.048\text{ m}$

Helix angle, $\alpha = 30°$

Number of turns, $n = 12$

Axial load, $W = 120\text{ N}$

Modulus of rigidity, $C = 80\text{ GN/m}^2$

Young's Modulus, $E = 200\text{ GN/m}^2.$

(i) Axial deflection, δ:

Axial deflection is given by :

$$\delta = 2\,WR^3\,n\,\pi\sec\alpha\left[\frac{\cos^2\alpha}{CI_p} + \frac{\sin^2\alpha}{EI}\right] \qquad \text{(Eq. 14·17)}$$

$$= 2 \times 120 \times (0.048)^3 \times 12 \times \pi \times \sec 30°$$

$$\left[\frac{(\cos 30°)^2}{80 \times 10^9 \times \frac{\pi}{32} \times (0.008)^4} + \frac{(\sin 30°)^2}{200 \times 10^9 \times \frac{\pi}{64} \times (0.008)^4}\right]$$

$$= 1.1554\,(0.0233 + 0.006217) = 0.0341\text{ m} = 34.1\text{ mm}$$

Hence, $\delta = \mathbf{34.1\ mm}$ **(Ans.)**

(ii) Angular rotation, ψ:

Angular rotation, $$\psi = 2WR^2\,n\,\pi\sin\alpha\left(\frac{1}{CI_p} - \frac{1}{EI}\right)$$

$$= 2 \times 120 \times (0.048)^2 \times 12 \times \pi \times \sin 30° \times$$

$$\left[\frac{1}{80 \times 10^9 \times \frac{\pi}{32} \times (0.008)^4} - \frac{1}{200 \times 10^9 \times \frac{\pi}{64}\,(0.008)^4}\right]$$

$$= 10.423\,(0.03108 - 0.02486) = 0.0648\text{ rad.} = 3.71°$$

Hence, $\psi = \mathbf{3.71°}$ **(Ans.)**

Example 14·16. *An open-coiled helical spring consists of 12 coils, each of mean diameter 60 mm, the wire forming the coil being 6 mm in diameter. Each coil makes an angle of 30° with the plane perpendicular to the axis of the spring.*

(i) Determine the load required to elongate the spring by 25 mm and the bending and shear stresses caused by that load;

(ii) Calculate the axial twist that would cause a bending stress of 50 MN/m² in the coils.

Take: $E = 200\ GN/m^2$, *and* $C = 82\ GN/m^2$.

Solution. Wire diameter, $d = 6\text{ mm} = 0.006\text{ m}$

Coil diameter (mean),

$D = 60$ mm $= 0{\cdot}06$ m

Number of turns, $n = 12$

Helix angle, $\alpha = 30^\circ$

Deflection $\delta = 25$ mm $= 0{\cdot}025$ m

$E = 200$ GN/m²

$C = 82$ GN/m².

(i) Load W, σ_b, τ :

Using the relation, $\delta = 2\,WR^3\,n\pi\,\sec\alpha\left(\dfrac{\cos^2\alpha}{CI_p} + \dfrac{\sin^2\alpha}{EI}\right)$ with usual notations (Eq. 14·17),

we have $0{\cdot}025 = 2W \times (0{\cdot}06/2)3 \times 12 \times \pi\sec 30^\circ \times$

$$\left[\frac{(\cos 30^\circ)^2}{80\times 10^9 \times \dfrac{\pi}{32}\times(0{\cdot}006)^4} + \frac{(\sin 30^\circ)^2}{200\times 10^9\times\dfrac{\pi}{64}\times(0{\cdot}006)^4}\right]$$

$$= 0{\cdot}00235\;W\,(0{\cdot}07188 + 0{\cdot}01965)$$

$\therefore$ $\quad$ **$W = 116$ N (Ans.)**

Bending moment, $M = WR\sin\alpha = 116\times(0{\cdot}06/2)\times\sin 30^\circ = 1{\cdot}74$ Nm

Now bending stress,

$$\sigma_b = \frac{M}{Z} = \frac{M}{I/y} = \frac{My}{I} = \frac{M\times d/2}{\dfrac{\pi}{64}\times d^4} = \frac{32\,M}{\pi d^3} \qquad \text{...}(i)$$

$$= \frac{32\times 1{\cdot}74}{\pi\times(0{\cdot}006)^3}\times 10^{-6}\ \text{MN/m}^2 = 82{\cdot}05\ \text{MN/m}^2$$

i.e. $\quad$ **$\sigma_b = 82{\cdot}05$ MN/m² (Ans.)**

Shock-absorbers on a special purpose motor vehicle.

Coil diameter (mean),

$D = 60 \text{ mm} = 0{\cdot}06 \text{ m}$

Number of turns, $n = 12$

Helix angle, $\alpha = 30°$

Deflection $\delta = 25 \text{ mm} = 0{\cdot}025 \text{ m}$

$E = 200 \text{ GN/m}^2$

$C = 82 \text{ GN/m}^2$.

(i) Load W, σ_b, τ :

Using the relation, $\delta = 2\,WR^3\,n\pi \sec\alpha \left(\dfrac{\cos^2\alpha}{CI_p} + \dfrac{\sin^2\alpha}{EI}\right)$ with usual notations (Eq. 14.17),

we have $0{\cdot}025 = 2W \times (0{\cdot}06/2)3 \times 12 \times \pi \sec 30° \times$

$$\left[\frac{(\cos 30°)^2}{80 \times 10^9 \times \dfrac{\pi}{32} \times (0{\cdot}006)^4} + \frac{(\sin 30°)^2}{200 \times 10^9 \times \dfrac{\pi}{64} \times (0{\cdot}006)^4}\right]$$

$$= 0{\cdot}00235\ W\,(0{\cdot}07188 + 0{\cdot}01965)$$

$\therefore$ $\quad$ **$W = 116$ N (Ans.)**

Bending moment, $M = WR \sin\alpha = 116 \times (0{\cdot}06/2) \times \sin 30° = 1{\cdot}74$ Nm

Now bending stress,

$$\sigma_b = \frac{M}{Z} = \frac{M}{I/y} = \frac{My}{I} = \frac{M \times d/2}{\dfrac{\pi}{64} \times d^4} = \frac{32\,M}{\pi d^3} \qquad ...(i)$$

$$= \frac{32 \times 1{\cdot}74}{\pi \times (0{\cdot}006)^3} \times 10^{-6}\ \text{MN/m}^2 = 82{\cdot}05\ \text{MN/m}^2$$

i.e. $\quad$ **$\sigma_b = 82{\cdot}05$ MN/m² (Ans.)**

Shock-absorbers on a special purpose motor vehicle.

Twisting moments about the axis of the spring,

$$T = WR \cos \alpha = 116 \times (0{\cdot}06/2) \times \cos 30^\circ = 3{\cdot}013 \text{ Nm}$$

But, $\dfrac{T}{I_p} = \dfrac{\tau}{r}$ $\quad \therefore \; \tau = \dfrac{Tr}{I_p} = \dfrac{T \times r}{\dfrac{\pi}{32} \times d^4} = \dfrac{T \times d}{2 \times \dfrac{\pi d^4}{32}} = \dfrac{16\,T}{\pi d^3}$

$$= \frac{16 \times 3{\cdot}013}{\pi \times (0{\cdot}006)^3} \times 10^{-6} = 71{\cdot}04 \text{ MN/m}^2$$

i.e. $\quad \boldsymbol{\tau = 71{\cdot}04}$ **MN/m²** **(Ans.)**

(*ii*) Axial twist, T':

Let, $\quad T'$ = Axial torque required to cause bending stress of 50 MN/m².

Component of axial torque causing bending = $T' \cos \alpha$

From eqn. (*i*), we have :

$$50 \times 10^6 = \frac{32\,T' \cos \alpha}{\pi d^3}$$

$$\therefore \quad T' = \frac{50 \times 10^6 \times \pi \times (0{\cdot}006)^3}{32 \times \cos 30^\circ}$$

Hence, $\quad \boldsymbol{T' = 1{\cdot}22}$ **Nm (Ans.)**

Example 14·17. *An open-coiled helical spring of wire diameter 12 mm, mean coil radius 84 mm, helix angle 20° carries an axial load of 480 N. Determine the shear stress and direct stress developed at inner radius of the coil.*

Solution. Diameter of wire,

$$d = 12 \text{ mm} = 0{\cdot}012 \text{ m}$$

Mean coil radius, $\quad R = 84 \text{ mm} = 0{\cdot}084 \text{ m}$

Helix angle, $\quad \alpha = 20^\circ$

Axial load, $\quad W = 480$ N.

Shear stress; direct stress:

Twisting moment, $\quad T = WR \cos \alpha$

Bending moment, $\quad M = WR \sin \alpha$

Torsional shear stress, $\quad \tau_1 = \dfrac{16\,T}{\pi d^3} = \dfrac{16 \times WR \cos \alpha}{\pi d^3}$

$$= \frac{16 \times 480 \times 0{\cdot}084 \times \cos 20^\circ}{\pi \times (0{\cdot}012)^3} \times 10^{-6} \text{ MN/m}^2 = 111{\cdot}66 \text{ MN/m}^2$$

Direct shear stress, $\quad \tau_2 = \dfrac{W}{\dfrac{\pi}{4} d^2} = \dfrac{4\,W}{\pi d^2} = \dfrac{4 \times 480}{\pi \times (0{\cdot}012)^2} \times 10^{-6} \text{ MN/m}^2 = 4{\cdot}24 \text{ MN/m}^2$

$\therefore$ Total shear stress at the inner coil radius,

$$\tau = \tau_1 + \tau_2 = 111{\cdot}66 + 4{\cdot}24 = 115{\cdot}9 \text{ MN/m}^2$$

Hence, $\quad \boldsymbol{\tau = 115{\cdot}9}$ **MN/m²** **(Ans.)**

Direct stress due to bending,

$$\sigma_b = \frac{32\,M}{\pi d^3} = \frac{32 \times WR \sin \alpha}{\pi d^3}$$

$$= \frac{32 \times 480 \times 0.084 \times \sin 20°}{\pi \times (0.012)^3} \times 10^{-6}\ \text{MN/m}^2 = 81.28\ \text{MN/m}^2$$

Hence, $\sigma_b = 81.28\ \text{MN/m}^2$ **(Ans.)**

Example 14·18. *A open-coiled helical spring made of steel wire 6 mm diameter and 30 mm mean coil radius, with 65° inclination of the coils with the spring axis, is subjected to an axial torque T. If number of turns in the spring increases by 1/8 and the original number of turns is 12 calculate:*

(i) Magnitude of axial torque T;

(ii) Change in axial length of the spring.

Take: $C_{steel} = 84\ GN/m^2$, *and* $E_{steel} = 210\ GN/mm^2$.

Solution. Diameter of steel wire,

$$d = 6\ \text{mm} = 0.006\ \text{m}$$

∴ Polar moment of inertia,

$$I_p = \frac{\pi}{32} \times d^4 = \frac{\pi}{32} \times (0.006)^4 = 1.272 \times 10^{-10}\ \text{m}^4$$

Moment of inertia, $I = \frac{I_p}{2} = 6.36 \times 10^{-11}\ \text{m}^4$

Mean radius of coil, $R = 30\ \text{mm} = 0.03\ \text{m}$

Angle of helix, $\alpha = 90 - 65 = 25°$

Number of turns, $n = 12$

Angular rotation, $\phi = \frac{1}{8}\ \text{turn} = \frac{1}{8} \times 360° = 45° = 0.7854$ radian

(i) Magnitude of axial torque, *T*:

Angular rotation, $\phi = 2\,TRn\,\pi \sec\alpha \left[\frac{\sin^2\alpha}{CI_p} + \frac{\cos^2\alpha}{EI}\right]$ ...(Eqn. 14·19)

$$\therefore 0.7854 = 2T \times 0.03 \times 12\,\pi \sec 25° \left[\frac{(\sin 25°)^2}{84 \times 10^9 \times 1.272 \times 10^{-10}} + \frac{(\cos 25°)^2}{210 \times 10^9 \times 6.36 \times 10^{-11}}\right]$$

$$= 2.495\ \text{T}\ (0.0167 + 0.0615) \qquad \therefore T = 4.33\ \text{Nm}$$

(ii) Change in axial length of spring, δ:

Change in axial length of spring is given by:

$$\delta = 2\,TR^2\,n\,\pi \sin\alpha \left(\frac{1}{CI_p} - \frac{1}{EI}\right)$$

$$= 2 \times 4.033 \times (0.03)^2 \times 12\,\pi \sin 25° \times \left[\frac{1}{84 \times 10^9 \times 1.272 \times 10^{-10}} - \frac{1}{210 \times 10^9 \times 6.36 \times 10^{-11}}\right]$$

$$= 0.1156\ (0.0936 - 0.0748) = 0.002173\ \text{m} = 2.173\ \text{mm}$$

Hence, $\delta = 2.173\ \text{mm}$ **(Ans.)**

14·5. SPRINGS IN SERIES

Fig. 14·4 shows two springs connected in series.

Let, W = Load applied, k_1 = Stiffness of spring 1,

k_2 = Stiffness of spring 2, δ_1 = Extension of spring 1,

δ_2 = Extension of spring 2, and k = Stiffness of composite spring.

It can be easily imagined that each spring will be subjected to load W and the total extension produced will be the sum of extensions of two springs.

$\therefore$ Total extension, $\delta = \delta_1 + \delta_2$

$$\therefore \quad \frac{W}{k} = \frac{W}{k_1} + \frac{W}{k_2} \qquad \left(\because \delta = \frac{W}{k}\right)$$

$$\text{or,} \quad \frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \qquad \text{...(14·26)}$$

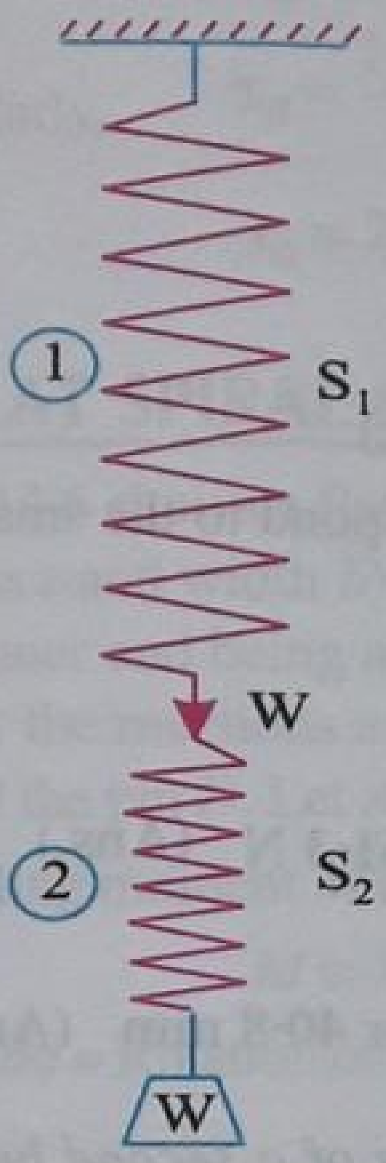


Fig. 14.4. Springs in series.

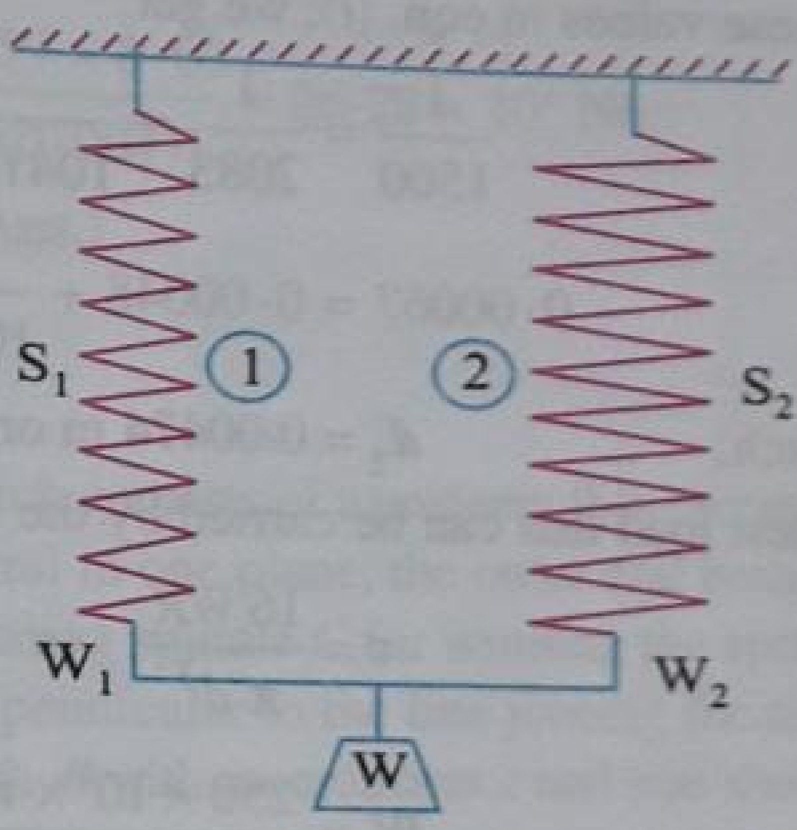


Fig. 14.5. Springs in parallel.

14·6. SPRINGS IN PARALLEL

Fig. 14·5 shows two springs, connected in parallel. When subjected to load, W, they will extend equally say by an amount δ. The load will be shared such that,

$$W = W_1 + W_2$$

$$\text{i.e.,} \quad \delta \cdot k = \delta \cdot k_1 + \delta \cdot k_2$$

$$\text{or,} \quad k = k_1 + k_2 \qquad \text{...(14·27)}$$

Suspension spring near the front wheel of an automobile.

Example 14·19. *A composite spring has two close-coiled springs connected in series; one spring has 12 coils of a mean diameter of 25 mm and wire diameter 2·5 mm. Find the wire diameter of the other spring, if it has 15 coils of mean diameter 40 mm. The stiffness of the composite spring is 1·5 kN/m.*

Determine the greatest load that can be carried by the composite spring and the corresponding extension if maximum stress is 250 MN/m². C = 80 GN/m².

Solution. In case of spring connected in series,

$$\delta = \delta_1 + \delta_2$$

$$\text{or,} \quad \frac{W}{k} = \frac{W}{k_1} + \frac{W}{k_2}$$

or, $$\frac{1}{k}=\frac{1}{k_1}+\frac{1}{k_2} \quad ...(i)$$

but, $$k_1=\frac{Cd_1^4}{64R_1^3\ n_1}=\frac{80\times10^9\times(0\cdot0025)^4}{64\times(0\cdot0125)^3\times12}=2083\text{ N/m}$$

$$k_2=\frac{Cd_2^4}{64\,R_2^3\ n_2}=\frac{80\times10^9\times d_2^4}{64\times(0\cdot02)^3\times15}=10416\times10^9\ d_2^4$$

Putting these values in eqn. (*i*), we get

$$\frac{1}{1500}=\frac{1}{2083}+\frac{1}{10416\times10^9 d_2^4}$$

$$0\cdot00067=0\cdot00048+\frac{1}{10416\times10^9 d_2^4}$$

From which, $d_2 = 0{\cdot}00474$ m or **4·74 mm** **(Ans.)**

The greatest load that can be carried by the spring will correspond to the smaller diameter.

$$\therefore \quad \tau=\frac{16\ WR}{\pi\ d^3},\ \text{ or }\ W=\frac{\tau\cdot\pi\ d^3}{16\ R}$$

$$W=\frac{250\times10^6\times\pi\times(0\cdot0025)^3}{16\times0\cdot0125}=\mathbf{61{\cdot}3\ N}\quad\textbf{(Ans.)}$$

Total extension, $$\delta=\frac{W.}{k}=\frac{61\cdot3}{1\cdot5\times1000}=0{\cdot}0408\text{ m or }\mathbf{40{\cdot}8\ mm}\quad\textbf{(Ans.)}$$

Example 14·20. *A helical spring B is placed inside the coils of a second helical spring A, having the same number of coils and free axial length and of same material. The two springs are compressed by an axial load of 210 N which is shared between them. The mean coil diameters of A and B are 90 mm and 60 mm and the wire diameters are 12 mm and 7 mm respectively. Calculate the load taken and the maximum stress in each spring.* **(I.E.S.)**

Solution. Let, W_A = Load shared by spring *A*,

W_B = Load shared by spring *B*,

δ_A = Deflection of spring *A*, and

δ_B = Deflection of spring *B*.

Now, $W_A + W_B = 210$ N .... (*Given*) ...(*i*)

Also, $\delta_A = \delta_B$

($\because$ Springs are connected in *parallel*)

$$\therefore \quad \frac{64\ W_A\ R_A^3\ n_A}{C_A\ d_A^4}=\frac{64\ W_B\ R_B^3\ n_B}{C_B\ d_B^4}$$

or, $$\frac{W_A\ R_A^3}{d_A^4}=\frac{W_B\ R_B^3}{d_B^4}\qquad\begin{bmatrix}\because n_A=n_B;\\ C_A=C_B\end{bmatrix}$$

or, $$\frac{W_A}{W_B}=\left[\frac{R_B}{R_A}\right]^3\cdot\left[\frac{d_A}{d_B}\right]^4,\quad\text{or,}\quad\frac{W_A}{W_B}=\left[\frac{30}{45}\right]^3\times\left[\frac{12}{7}\right]^4$$

or, $$\frac{W_A}{W_B}=2\cdot559\quad\text{or,}\quad W_A=2\cdot559\ W_B \quad ...(ii)$$

Substituting this value of W_A in (i) we get

$$2.559\, W_B + W_B = 210$$

$$W_B = 59\ N \quad \textbf{(Ans.)}$$

and, $\quad W_A = 210 - 59 = \mathbf{151\ N}$ **(Ans.)**

Now, $$\tau_A = \frac{16\, W_A\, R_A}{\pi\, d_A^3} = \frac{16 \times 151 \times 0.045}{\pi \times (0.012)^3} = 20 \times 10^6\ \text{N/m}^2$$

or, $\quad \tau_A = \mathbf{20\ MN/m^2}$ **(Ans.)**

Similarly, $$\tau_B = \frac{16\, W_B\, R_B}{\pi\, d_B^3} = \frac{16 \times 59 \times 0.03}{\pi \times (0.007)^3} = 26.28 \times 10^6\ \text{N/m}^2$$

or, $\quad \tau_B = \mathbf{26.28\ MN/m^2}$ **(Ans.)**

14·7. FLAT SPIRAL SPRING

Fig. 14·6 shows a flat spiral spring which consists of a uniform thin rectangular metallic strip (of thickness t and width b) wound into a spiral in one plane, the outer end being anchored to a pin D and the inner end being attached to the winding spindle C for winding the spring. Let H and V be respectively the reactions at D along and perpendicular to the line joining the axis of the spindle to the centre of the pin D. Let AB be a small length dl with co-ordinates x and y as shown in Fig. 14·6.

Bending moment M at the element (taken positive if the number of turns increase) is given by

$$M = V \cdot x - H \cdot y \qquad \text{...(14·28)}$$

Let, $d\phi$ = Rotation of the end B with respect to end A of the element dl due to the bending moment M,

Then, $$d\phi = \frac{M\ dl}{EI}$$

or, $$d\phi = \frac{(V \cdot x - H \cdot y)}{EI} \cdot dl \qquad \text{(from eqn. 14·28)}$$

Integrating both sides, we get

$$\phi = \frac{V}{EI} \int x\ dl - \frac{H}{EI} \int y\ dl \qquad \text{...(14·29)}$$

If the centroid of the spring is assumed to be at the centre of the spindle C, the first moment of the length l about the line CD will be zero.

i.e. $$\int y\ dl = 0 \qquad \text{...[14·30 (a)]}$$

and, moment of the length about YY line gives

$$\int x\ dl = l\ R \qquad \text{...[14·30 (b)]}$$

where, l = Length of spring strip, and

R = The distance between points C and D.

Also, the spring strip will be in equilibrium, when

$$T = V \cdot R \qquad \text{...(14·31)}$$

(where, T = winding torque)

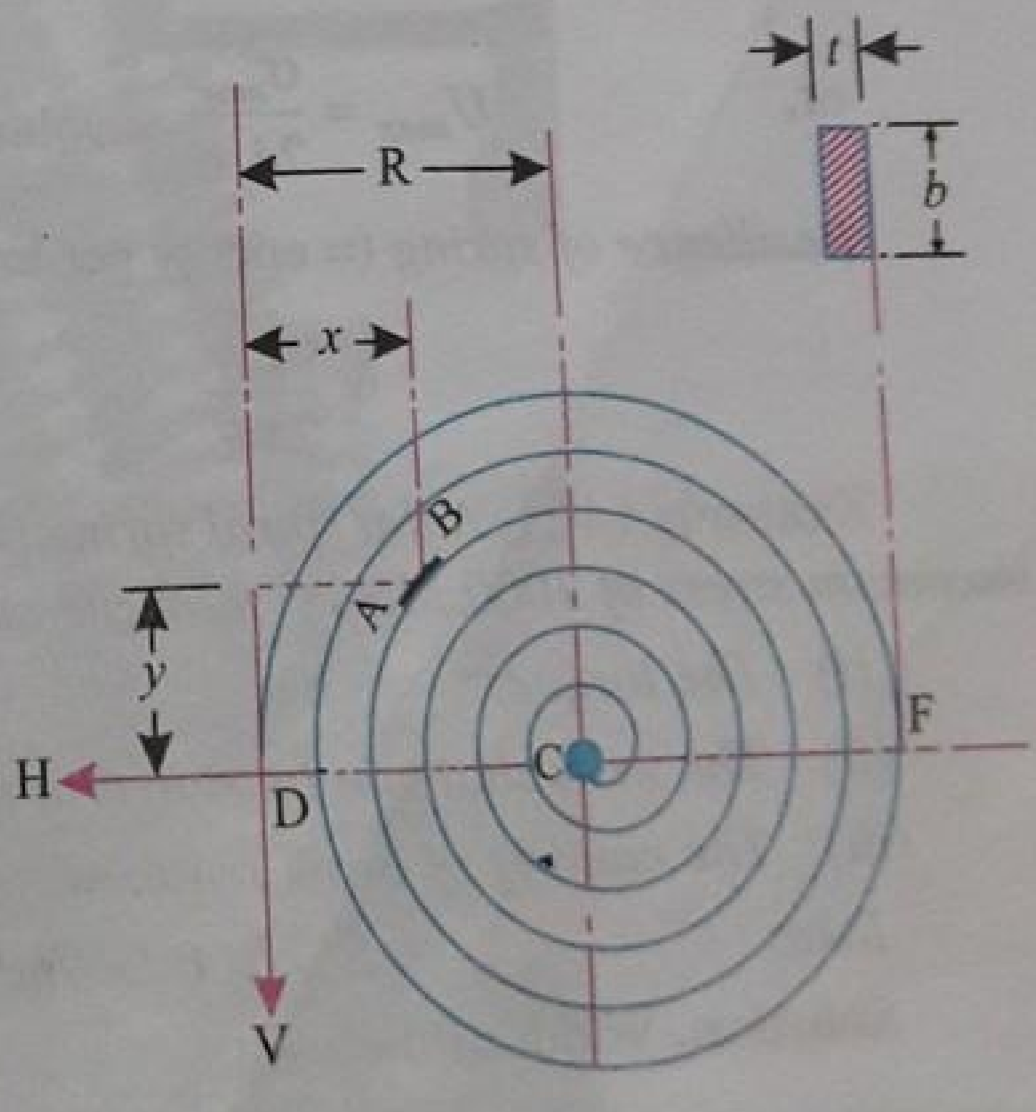


Fig. 14.6

Substituting eqns. (14·30) and (14·31) in eqn. (14·29), we have

$$\phi = \frac{(T/R)}{EI} \cdot (l \cdot R) = \frac{Tl}{EI} \quad \text{...(14·32)}$$

From eqn. (14·28) we find that the maximum bending moment will be obtained on such a section of strip where x *is maximum and* y *is minimum*. Obviously the point F is such a point where y is zero and x is *maximum*.

Let, $DF \simeq 2R$

Then from eqn. (14·31), we get

$$M_{max} = M_F = \frac{T}{R} \cdot (2R) = 2T \quad \text{...(14·33)}$$

The corresponding maximum bending stress on the cross-section of the strip,

$$\sigma_{max} = \frac{2T}{I} \times \frac{t}{2} = \frac{12\,T\,t}{bt^3} = \frac{12\,T}{bt^2} \quad \text{...[14·34 (a)]}$$

or,

$$T = \frac{bt^2 \cdot \sigma_{max}}{12} \quad \text{...[14·34 (b)]}$$

[If σ_{max} is known, the value of T can be calculated]

Energy stored (U):

The energy stored (corresponding to the torque T) can be calculated as follows:

We know,

$$U = \frac{1}{2} T\,\phi = \frac{1}{2} T \cdot \frac{Tl}{EI} = \frac{T^2 l}{2\,EI} = \frac{T^2 l}{2E \times \dfrac{bt^3}{12}} = \frac{6\,T^2 l}{E\,bt^3}$$

i.e.

$$U = \frac{6\,T^2 l}{E\,b\,t^3} \quad \text{...(14·35)}$$

To find maximum energy which can be stored (when the stress reaches its maximum value) will be obtained by substituting the value of T from eqn. [14·34 (*b*)] in eqn. (14·35).

$$\therefore \quad U_{max} = \frac{6\,l}{E\,bt^3}\left[\frac{bt^2\,\sigma_{max}}{12}\right]^2 = \frac{\sigma_{max}^2}{24\,E}\,(btl)$$

or,

$$U_{max} = \frac{\sigma_{max}^2}{24\,E} \times \text{volume of spring} \quad \text{...(14·36)}$$

Resilience of spring (= energy per unit volume)

$$= \frac{\sigma_{max}^2}{24\,E} \quad \text{...(14·37)}$$

Example 14·21. *A flat spiral spring is 5 mm wide, 0·25 mm thick and 3 metres long. Assuming maximum stress of 1000 MN/m² to occur at the point of greatest bending moment, calculate:*

(*i*) *The torque;*

(*ii*) *The work that can be stored in the spring; and*

(*iii*) *The number of turns required to wind up the spring.*

Take: $E = 200$ *GN/m²*.

(Madras University)

Solution. Width of the strip,

$$b = 5 \text{ mm}$$

Thickness of the strip, $t = 0·25$ mm

Length of the strip, $l = 3$ m

Maximum stress, $\sigma_{max} = 1000$ MN/m^2 (or 1000 N/mm^2).

(i) The torque T:

Now torque, $T = \dfrac{bt^2 \cdot \sigma_{max}}{12}$ with usual notations ...[Eqn. 14·34 (*b*)]

$$= \frac{5 \times 0 \cdot 25^2 \times 1000}{12} = 26 \text{ Nmm} = 0 \cdot 026 \text{ Nm}$$

i.e. $T = 0{\cdot}026$ **Nm (Ans.)**

(ii) Work that can be stored, U:

$$U = \frac{\sigma_{max}^2}{24\,E} \times \text{volume of spring} \qquad \text{...(Eqn. 14·36)}$$

$$= \frac{(1000 \times 10^6)^2}{24 \times 200 \times 10^9} \times [(5 \times 10^{-3}) \times (0 \cdot 25 \times 10^{-3} \times 3] = 0 \cdot 781 \text{ Nm}$$

i.e. $U = 0{\cdot}781$ **Nm (or J) (Ans.)**

(iii) Number of turns:

We know, $\phi = \dfrac{Tl}{EI}$ ...(Eqn. 14·32)

$$= \frac{0 \cdot 026 \times 3}{200 \times 10^9 \times \dfrac{bt^3}{12}}$$

$$= \frac{0 \cdot 026 \times 3 \times 12}{200 \times 10^9 \times 0 \cdot 005 \times (0 \cdot 00025)^3} = 59 \cdot 9 \text{ radians}$$

Shock-absorber system.

Since, 1 turn = 2π radians,

$$\therefore \textit{Number of turns} = \frac{59 \cdot 9}{2\pi} = 9 \cdot 533 \text{ turns} \quad \textbf{(Ans.)}$$

Example 14·22. *A flat spiral spring is made of a strip 6 mm wide, 0·25 mm thick and 12 m long. The torque is applied at the winding spindle and 9 complete turns are given. Calculate:*

(i) The torque;

(ii) Maximum stress developed at the point of greatest bending moment;

(iii) The energy stored.

Take: $E = 210 \text{ GN/m}^2$.

Solution. Width of the strip,

$$b = 6 \text{ mm } (= 0{\cdot}006 \text{ m})$$

Thickness of the strip, $t = 0{\cdot}25$ mm $(= 0{\cdot}00025$ m)

Length of the strip, $l = 12$ m

Number of complete turns, $n = 9$

Young's modulus, $E = 210 \text{ GN/m}^2$.

(i) The torque, *T*:

Angular rotation,

$$\phi = 2\pi n = 2\pi \times 9 = 56{\cdot}55 \text{ radians}$$

Also, $$\phi = \frac{Tl}{EI} \quad \therefore T = \frac{\phi EI}{l} = \frac{56 \cdot 55 \times 210 \times 10^9 \times (bt^3/12)}{l}$$

$$= \frac{56 \cdot 55 \times 210 \times 10^9 \times (0 \cdot 006) \times (0 \cdot 00025)^3}{12 \times 12} \text{ Nm}$$

$$= 0{\cdot}00773 \text{ Nm} = 7{\cdot}73 \text{ Nmm.}$$

i.e. $\quad T = \mathbf{7{\cdot}73 \text{ Nmm}}$ **(Ans.)**

(ii) Maximum stress, σ_{max}:

$$\sigma_{max} = \frac{12\,T}{bt^2} = \frac{12 \times 0 \cdot 00773}{0 \cdot 006 \times (0 \cdot 00025)^2} \times 10^{-6} \text{ MN/m}^2 = 247 \cdot 4 \text{ MN/m}^2$$

[Eqn. 14·34 (*a*)]

i.e. $\quad \sigma_{max} = \mathbf{247{\cdot}4 \text{ MN/m}^2}$ **(Ans.)**

(iii) The energy stored, *U*:

$$U = \frac{1}{2} T\,\phi = \frac{1}{2} \times 7 \cdot 73 \times 56 \cdot 55$$

$$= 218.56 \text{ Nmm}$$

i.e. $\quad U = \mathbf{218{\cdot}56 \text{ Nmm}}$ **(Ans.)**

14·8. LAMINATED SPRINGS

These springs are called *semielliptical, leaf or carriage springs* and find their use in *trucks, trains, trolleys* etc. They consist of a number of leaves of *spring steel* held together at the centre with clamps. The plates are provided with curvature initially and the ends of the top plate are pin jointed to chassis of the vehicle. (Fig. 14·7). The load at which the plates become straight is called "*Proof load*".

14·8·1. Semi-elliptical Spring:

Refer to Fig. 14·7.

Let, $\quad b$ = Width of each plate,

$\quad t$ = Thickness of each plate,

a = Overlap at each end,
N = Number of plates in the spring,
l = The spring span length,
σ_b = Bending stress, and
E = Young's modulus of elasticity.

The maximum bending moment $\frac{Wl}{4}$ occurs at the centre and is resisted by all N plates equally.

$\therefore$ Resisting moment of each plate,

$$M = \frac{W\,l}{4\,N}$$

Now, $$\sigma_b = \frac{My}{I}$$

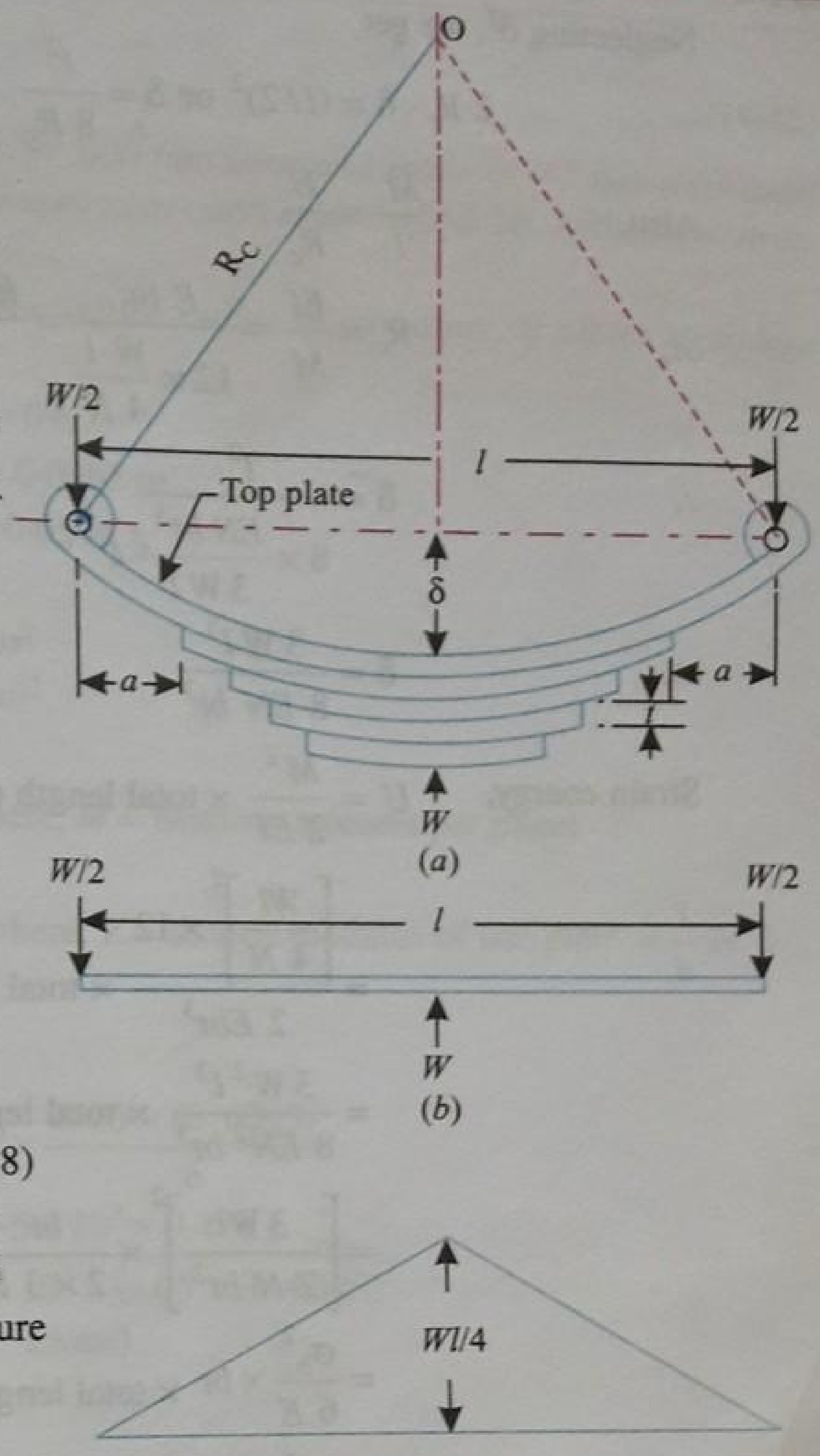
or, $$\sigma_b = \frac{W\,l}{4\,N} \cdot \frac{t}{2} \cdot \frac{1}{\frac{bt^3}{12}}$$

or, $$\sigma_b = \frac{3\,W\,l}{2\,Nbt^2} \qquad \text{...(14·38)}$$

Let, δ = Initial deflection of the plates, and
R_c = Radius of curvature of the plates.

Refer to Fig. 14·7.

$$R_c^{\,2} = (l/2)^2 + (R_c - \delta)^2$$
$$R_c^{\,2} = (l/2)^2 + R_c^{\,2} + \delta^2 - 2\,R_c\delta$$



(c) B.M. diagram
Fig. 14.7


A tunnel making machine is being transported. Heavy vehicles use leaf springs to bear loads and shocks.

Neglecting δ^2, we get

$$2\,R_c \cdot \delta = (l/2)^2 \text{ or } \delta = \frac{l^2}{8\,R_c}$$

Also,
$$\frac{M}{I} = \frac{E}{R_c}$$

or,
$$R_c = \frac{EI}{M} = \frac{E\,bt^3}{12 \times \frac{W\,l}{4\,N}} = \frac{E\,N\,bt^3}{3\,Wl}$$

$\therefore$
$$\delta = \frac{l^2}{8 \times \frac{EN\,bt^3}{3\,W\,l}} \; Rc$$

or,
$$\delta = \frac{3\,W\,l^3}{8\,EN\,bt^3} \qquad \text{...(14·39)}$$

Strain energy,
$$U = \frac{M^2}{2\,EI} \times \text{total length of leaves}$$

$$= \frac{\left[\frac{Wl}{4\,N}\right]^2 \times 12}{2\,Ebt^3} \times \text{total length of leaves}$$

$$= \frac{3\,W^2\,l^2}{8\,EN^2\,bt^3} \times \text{total length of leaves}$$

$$= \left[\frac{3\,Wl}{2\,N\,bt^2}\right]^2 \times \frac{bt}{2 \times 3\,E} \times \text{total length of leaves}$$

$$= \frac{\sigma_b^{\,2}}{6\,E} \times bt \times \text{total length of leaves} = \frac{\sigma_b^{\,2}}{6\,E} \times \text{volume of spring}$$

i.e.
$$U = \frac{\sigma_b^{\,2}}{6\,E} \times \text{volume of spring} \qquad \text{...(14·40)}$$

14·8·2. Quarter Elliptical Spring

These springs are called *cantilever laminated springs*. In Fig. 14·8 is shown a spring of this type.

It has an effective span length l and carries a load W. If observed carefully it will be noted that the spring is equivalent to a laminated semi-elliptical type spring of length $2l$ carrying a load $2W$ at the centre. Hence by replacing l by $2l$ and W by $2W$ the deflection and stress can be obtained as follows:

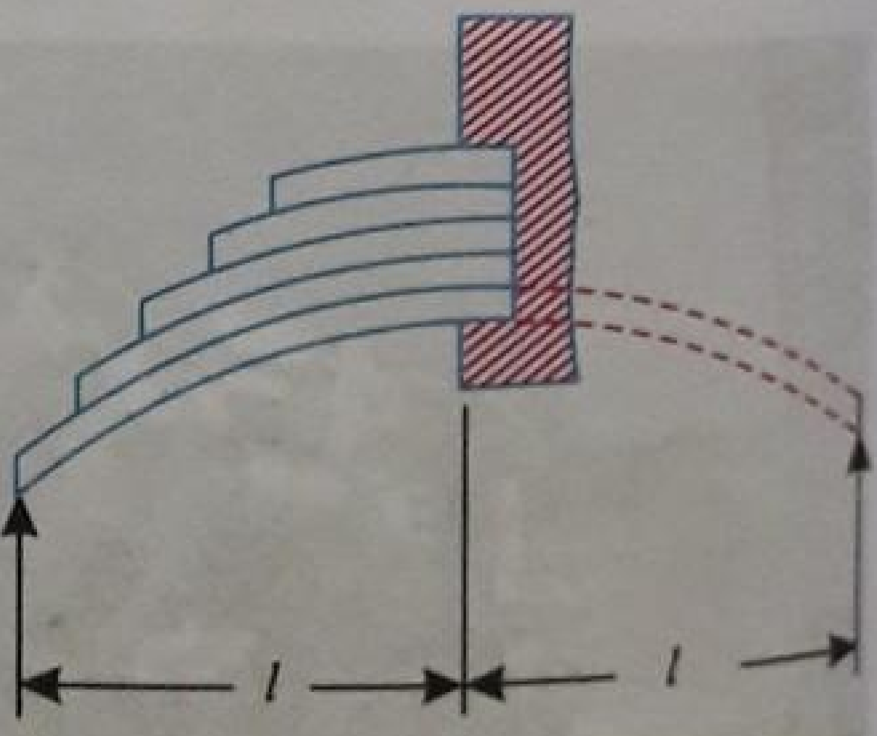


Fig. 14.8

$$\delta = \frac{3\,(2W) \cdot (2l)^3}{8\,EN\,bt^3}$$

or,
$$\delta = \frac{6\,Wl^3}{EN\,bt^3} \qquad \text{...(14·41)}$$

Similarly,
$$\sigma_b = \frac{3\,(2W)\,(2l)}{2\,N\,bt^2}$$

$$\therefore \qquad \sigma_b = \frac{6\,Wl}{N\,bt^2} \qquad ...(14\text{-}42)$$

Example 14·23. *A carriage spring is to be 600 mm long and made of 9·5 mm thick steel plates and 50 mm broad. How many plates are required to carry a load of 4·5 kN, without the stress exceeding 230 MN/m².*

What would be central deflection and the initial radius of curvature, if plates straighten under the load? E = 200 GN/m²

Solution. Span length, $l = 600 \text{ mm} = 0{\cdot}6 \text{ m}$

Thickness of each plate, $t = 9{\cdot}5 \text{ mm} = 0{\cdot}0095 \text{ m}$

Width of each plate, $b = 50 \text{ mm} = 0{\cdot}05 \text{ m}$

Load, $W = 4{\cdot}5 \text{ kN}$

$E = 200 \text{ GN/m}^2$

$\sigma_b = 230 \text{ MN/m}^2$.

Number of plates, N:

We know, $M = \dfrac{Wl}{4N}$ (where, M = bending moment per plate)

but, $M = \sigma_b \times Z$ (where, Z = section modulus of one plate $= \dfrac{1}{6}bt^2$)

$$\therefore \qquad \frac{Wl}{4N} = \sigma_b \times \frac{bt^2}{6}$$

$$\text{or,} \qquad \frac{4{\cdot}5 \times 10^3 \times 0{\cdot}6}{4 \times N} = 230 \times 10^6 \times \frac{0{\cdot}05 \times (0{\cdot}0095)^2}{6}$$

$$\therefore \qquad N = \frac{4{\cdot}5 \times 10^3 \times 0{\cdot}6 \times 6}{4 \times 230 \times 10^6 \times 0{\cdot}05 \times (0{\cdot}0095)^2}$$

or, $N = 3{\cdot}9$ **say 4 (Ans.)**

Initial radius of carvature, R_c:

Using the relation $\dfrac{\sigma_b}{y} = \dfrac{E}{R_c}$, we get

$$\frac{230 \times 10^6}{t/2} = \frac{200 \times 10^9}{R_c}$$

$$\therefore \qquad R_c = \frac{200 \times 10^9 \times 0{\cdot}0095}{230 \times 10^6 \times 2} = 4{\cdot}13 \text{ m} \quad \textbf{(Ans.)}$$

Central deflection, δ:

Using the relation: $\delta = \dfrac{3\,Wl^3}{8\,EN\,bt^3}$, we have

$$\delta = \frac{3 \times 4{\cdot}5 \times 10^3 \times (0{\cdot}6)^3}{8 \times 200 \times 10^9 \times 4 \times 0{\cdot}05 \times (0{\cdot}0095)^3} = 10{\cdot}6 \text{ mm} \quad \textbf{(Ans.)}$$

Example 14·24. *A laminated steel spring 1 m long is to support central load of 5·8 kN. If the maximum deflection of spring is not to exceed 45 mm and maximum stress should not exceed 300 MN/m², calculate:*

(i) *The thickness of the leaves;*

(ii) *Their number if each plate is to be 80 mm wide.*

Take: $E = 200 \text{ GN/m}^2$.

Solution. Span length, $l = 1\text{ m}$

Width of each plate, $b = 80\text{ mm} = 0{\cdot}08\text{ m}$

Load, $W = 5{\cdot}8\text{ kN}$

Deflection, $\delta = 45\text{ mm} = 0{\cdot}045\text{ m}$

Stress, $\sigma_b = 300\text{ MN/m}^2$

(*i*) Thickness, *t*:

(*ii*) Number of plates, *N*:

We know that,
$$\sigma_b = \frac{3}{2} \times \frac{W\,l}{N\,bt^2}$$
$$300 \times 10^6 = \frac{3 \times 5{\cdot}8 \times 10^3 \times 1}{2 \times N \times 0{\cdot}08\, t^2}$$

or,
$$Nt^2 = \frac{3 \times 5.8 \times 10^3 \times 1}{300 \times 10^6 \times 2 \times 0{\cdot}08} = 3{\cdot}625 \times 10^{-4} \qquad \text{...}(i)$$

Also,
$$\delta = \frac{3\,Wl^3}{8\,EN\,bt^3}$$
$$0{\cdot}045 = \frac{3 \times 5{\cdot}8 \times 10^3 \times 1^3}{8 \times 200 \times 10^9 \times N \times 0{\cdot}08 \times t^3}$$
$$Nt^3 = \frac{3 \times 5{\cdot}8 \times 10^3 \times 1^3}{0{\cdot}045 \times 8 \times 200 \times 10^9 \times 0{\cdot}08} = 3{\cdot}02 \times 10^{-6} \qquad \text{...}(ii)$$

Dividing (*ii*) by (*i*), we get
$$\frac{Nt^3}{Nt^2} = \frac{3{\cdot}02 \times 10^{-6}}{3{\cdot}625 \times 10^{-4}}$$

∴ $t = 0{\cdot}00833\text{ m} = \mathbf{8{\cdot}33\text{ mm}}$ **(Ans.)**

and, $Nt^3 = 3{\cdot}02 \times 10^{-6}$

or,
$$N = \frac{3{\cdot}02}{t^3} = \frac{3{\cdot}02 \times 10^{-6}}{(0{\cdot}00833)^3}$$

or, $N = 5{\cdot}22$ say **5** **(Ans.)**

Example 14·25. *A leaf spring of semi-elliptic type has 11 plates each 9 cm wide and 1·5 cm thick. The length of spring is 1·5 m. The plates are made of steel having a proof stress (bending) of 650 MN/m². To what radius should the plates be bent initially?*

From what height can a load of 600 N fall on to centre of the spring, if maximum stress is to be one-half of the proof stress?

$E = 200\text{ GN/m}^2$. **(Bombay University)**

Solution. The bending equation is given by,
$$\frac{M}{I} = \frac{\sigma_b}{y} = \frac{E}{R_c}$$
$$R_c = \frac{E \times y}{\sigma_b} \qquad \text{(where, } R_c = \text{radius of curvature)}$$
$$= \frac{E \times t/2}{\sigma_b} = \frac{200 \times 10^9 \times (0{\cdot}75/100)}{650 \times 10^6}$$

or, $R_c = \mathbf{2{\cdot}31\text{ m}}$ **(Ans.)**

The stress in the second case is *half* the proof stress

i.e. $\frac{650}{2} = 325 \text{ MN/m}^2$

Let, W = The equivalent static load which will produce this stress.

Then, $\sigma_b = \frac{3\,W\,l}{2\,N\,bt^2}$

$$W = \frac{2\,N\,bt^2\,\sigma_b}{3\,l} = \frac{2 \times 11 \times (9/100) \times (1\cdot 5/100)^2 \times 325 \times 10^6}{3 \times 1\cdot 5}$$

$= 32175$ N or $32\cdot 175$ kN

Deflection under this load, $\delta = \frac{3\,W\,l^3}{8\,EN\,bt^3} = \frac{3 \times 32175 \times (1\cdot 5)^3}{8 \times 200 \times 10^9 \times 11 \times (9/100) \times (1\cdot 5/100)^3}$

$= 0\cdot 061$ m $= 61$ mm

Let, P = Falling weight

Then, work done by the falling weight

$= P\,(h + \delta)$

where, h = Height through which the weight falls.

But, $P\,(h + \delta)$ = Energy stored in the spring due to static load,

$\therefore$ $P\,(h + \delta) = \frac{1}{2}\,W\delta$

$$600\,(h + 0\cdot 061) = \frac{1}{2} \times 32175 \times 0\cdot 061$$

$h = 1\cdot 574$ **m (Ans.)**

Leaf spring.

Example 14·26. *A quarter elliptical spring has a length of 50 cm and consists of plates each 6 cm wide and 0·6 cm thick. Find the least number of plates which can be used if deflection under gradually applied load of 3 kN is not to exceed 8 cm. E = 200 GN/m².*

Solution. Span length, $l = 50$ cm $= 0\cdot 5$ m

Width of each plate, $b = 6$ cm $= 0\cdot 06$ m

Thickness of each plate, $t = 0\cdot 6$ cm $= 0\cdot 006$ m

Deflection, $\delta = 8$ cm $= 0\cdot 08$ m

Load, $W = 3$ kN

Least number of plates, N:

Using the relation: $\delta = \frac{6\,W\,l^3}{EN\,bt^3}$, we have

$$0 \cdot 08 = \frac{6 \times 3 \times 10^3 \times (0 \cdot 5)^3}{200 \times 10^9 \times N \times 0 \cdot 06 \times (0 \cdot 006)^3}$$

$$\therefore \quad N = \frac{6 \times 3 \times 10^3 \times (0 \cdot 5)^3}{200 \times 10^9 \times 0 \cdot 08 \times 0 \cdot 06 \times (0 \cdot 006)^3}$$

$$N = 10 \cdot 85 \text{ say } \mathbf{11} \quad \textbf{(Ans.)}$$

Example 14·27. *A quarter elliptical leaf spring has a length of 600 mm and consists of plates each 50 mm wide and 6 mm thick.*

(i) Determine the least number of plates which can be used if the deflection under a gradually applied load of 1·8 kN is not to exceed 80 mm.

(ii) If the applied load of 1·8 kN, instead of being gradually applied, falls a distance of 6 mm on to the undeflected spring, find the maximum deflection and stress produced.

Take: $E = 200 \text{ GN/m}^2$.

Leaf spring fatigue testing system.

Solution. Length of the spring,

$$l = 600 \text{ mm} = 0 \cdot 6 \text{ m}$$

Width of each plate, $b = 50 \text{ mm} = 0 \cdot 05 \text{ m}$

Thickness of each plate, $t = 6 \text{ mm} = 0 \cdot 006 \text{ m}$

Gradually applied load, $W = 1 \cdot 8 \text{ kN}$

Deflection under the above load,

$$\delta = 80 \text{ mm} = 0 \cdot 08 \text{ m}$$

(i) Number of plates, N:

We know,
$$\delta = \frac{6 W l^3}{E N b t^3}$$

or,
$$0 \cdot 08 = \frac{6 \times 1 \cdot 8 \times 10^3 \times 0 \cdot 6^3}{200 \times 10^9 N \times 0 \cdot 05 \times (0 \cdot 006)^3}$$

or,
$$N = \frac{6 \times 1 \cdot 8 \times 10^3 \times 0 \cdot 6^3}{0 \cdot 08 \times 200 \times 10^9 \times 0 \cdot 05 \times (0 \cdot 006)^3} = 13 \cdot 5 \text{ say } 14$$

Hence, number of plates $= \mathbf{14}$ **(Ans.).**

(ii) δ' ; σ_{max}:

Let, W_e = Equivalent gradually applied load which would produce the same deflection as is caused by the impact load.

Then,
$$\delta' = \frac{6 W_e l^3}{EN b t^3} = \frac{6 W_e \times 0 \cdot 6^3}{200 \times 10^9 \times 14 \times 0 \cdot 05 \times (0 \cdot 006)^3}$$

or,
$$W_e = 23{\cdot}33 \times 10^3 \, \delta' \text{ N}$$

Loss of potential energy $= 1{\cdot}8 \times 10^3 \, (6 \times 10^{-3} + \delta')$

Strain energy absorbed by the spring

$$= \frac{1}{2} \cdot W_e \cdot \delta' = \frac{1}{2} \times 23 \cdot 33 \times 10^3 \times \delta'^2$$

$$\therefore \quad 1 \cdot 8 \times 10^3 \, (0 \cdot 006 + \delta') = \frac{1}{2} \times 23 \cdot 33 \times 10^3 \times \delta'^2$$

or,
$$0{\cdot}006 + \delta' = 6{\cdot}48 \, \delta'^2 \quad \text{or} \quad \delta'^2 - 0{\cdot}154 \, \delta' - 0{\cdot}000926 = 0$$

$$\therefore \quad \delta' = \frac{0 \cdot 154 \pm \sqrt{(0 \cdot 154)^2 + 4 \times 0 \cdot 000926}}{2}$$

$$= \frac{0 \cdot 154 \pm 0 \cdot 1656}{2} = 0 \cdot 1598 \text{ m} = 159 \cdot 8 \text{ mm}$$

Hence, deflection $= \mathbf{159 \cdot 8\ mm}$ **(Ans.)**

$\therefore$ $W_e = 23 \cdot 33 \times 10^3 \times 0 \cdot 0598 \times 10^{-3}$ kN $= 3 \cdot 728$ kN

Maximum stress, $\sigma_{max} = \frac{6\, W_e\, l}{N\, bt^2} = \frac{6 \times 3 \cdot 728 \times 10^3 \times 0 \cdot 6}{14 \times 0 \cdot 05 \times (0 \cdot 006)^2} \times 10^{-6}$ MN/m^2

$= 532 \cdot 57$ MN/m^2

Hence, $\sigma_{max} = \mathbf{532 \cdot 57\ MN/m^2}$ **(Ans.)**

TYPICAL EXAMPLES (For Competitive Examinations)

Example 14·28. *A close-coiled helical spring has stiffness of 10 N/mm. Its length when fully compressed with adjacent coils touching each other is 400 mm. The modulus of rigidity of material of the spring = 80000 N/mm².*

(i) *Determine the wire diameter and mean coil diameter, if their ratio is 1/10.*

(ii) *If the gap between any two adjacent coils is 2 mm, what maximum load can be applied before the spring becomes solid i.e. adjacent coils touch.*

(iii) *What is the corresponding maximum shear stress in the spring?*

Solution. Let, d = Wire diameter, mm,

D = Mean coil diameter, mm, and

n = No. of turns of the coil.

$$\frac{d}{D} = \frac{1}{10} \quad \text{..... (Given)}$$

i.e. $D = 10\, d$, and $R = 5\, d$

(i) Wire diameter, d:

When the coils are touching, length of spring,

$$400 = nd \qquad \therefore \quad n = \frac{400}{d}$$

Using the relation: $\delta = \frac{64\, WR^3 n}{Cd^4}$, we get

$$\frac{W}{\delta} = \frac{Cd^4}{64\, WR^3 n}$$

$$10 = \frac{80000 \times d^4}{64 \times (5d)^3 \times \frac{400}{d}}, \quad \text{or,} \quad d^2 = \frac{10 \times 64 \times 125 \times 400}{80000}$$

$\therefore$ $d = \mathbf{20\ mm}$ **(Ans.)**

Mean coil diameter, D:

$D = 10\, d = 10 \times 20 = \mathbf{200\ mm}$ **(Ans.)**

(ii) Load that can make the spring solid, W:

Difference between any two adjacent coils = 2 mm

No. of coils, $n = \frac{400}{d} = \frac{400}{20} = 20$

$\therefore$ Total deflection of the load

$= 20 \times 2 = 40$ mm

and, load that can make the spring solid,

$$W = k \cdot \delta \qquad \text{(where, } k = \text{stiffness)}$$

$$= 10 \times 40 \quad \text{or} \quad W = \mathbf{400\ N} \quad \textbf{(Ans.)}$$

(iii) **Maximum shear stress in the spring, τ :**

Using the relation: $\tau = \dfrac{16\,WR}{\pi d^3}$, we get

$$\tau = \frac{16 \times 400 \times 100}{\pi \times 20^3} = \mathbf{25 \cdot 46\ N/mm^2} \quad \textbf{(Ans.)}$$

Example 14·29. *A close-coiled helical spring is made of a round wire having n turns and the mean coil radius R is 5 times the wire diameter. Show that the stiffness of such a spring is $\frac{R}{n}$ × constant. Determine the constant when the modulus of rigidity C of the spring wire is 82000 N/mm².*

If the above spring is to support a load of 1·2 kN with 120 mm compression and the maximum shear stress 250 N/mm², calculate:

(i) Mean radius of the coil;

(ii) Number of turns;

(iii) Weight of the spring.

Assume density of material as 76·5 kN/m³.

Solution. Let, d = Diameter of the spring wire.

Then, mean radius of the coil,

$$R = 5d$$

$$\therefore \quad d = 0 \cdot 2\,R$$

Number of turns = n

Stiffness of the spring, $k = \dfrac{W}{\delta} = \dfrac{Cd^4}{64\,nR^3}$

$$= \frac{C \times (0 \cdot 2\ R)^4}{64\,nR^3} = \frac{C \times 1 \cdot 6 \times 10^{-3} \times R^4}{64\,nR^3} = \frac{R}{n} \times \text{constant}$$

Hence, *stiffness* $= \dfrac{R}{n} \times \text{constant}$ ......**(Proved)**

where, constant $= \dfrac{C \times 1 \cdot 6 \times 10^{-3}}{64} = \dfrac{82000 \times 1 \cdot 6 \times 10^{-3}}{64} = 2 \cdot 05$

∴ Stiffness, $k = 2 \cdot 05 \times \dfrac{R}{n}$

Another view of leaf spring.

Axial load on the spring,

$$W = 1{\cdot}2 \times 10^3 = 1200 \text{ N}$$

Compression,

$$\delta = 120 \text{ mm}$$

(i) Mean radius of the coil, R:

Stiffness,

$$k = 2{\cdot}05 \times \frac{R}{n}$$

$\therefore$

$$\frac{W}{\delta} = 2{\cdot}05 \times \frac{R}{n} \quad \text{or} \quad \frac{1200}{120} = 2{\cdot}05 \times \frac{R}{n}$$

or,

$$\frac{R}{n} = \frac{10}{2{\cdot}05}$$

Also, shear stress in the wire,

$$\tau = \frac{16\,WR}{\pi d^3}$$

But,

$$\tau = 250 \text{ N/mm}^2 \quad \text{...(Given)}$$

$\therefore$

$$250 = \frac{16 \times 1200 \times 5d}{\pi\, d^3} \quad \text{or} \quad d^2 = \frac{16 \times 1200 \times 5}{250 \times \pi} = 122{\cdot}23$$

$\therefore$

$$d = 11 \text{ mm}$$

and,

$$R = 11 \times 5 = 55 \text{ mm} \quad \textbf{(Ans.)}$$

(ii) Number of turns, n:

Now,

$$\frac{R}{n} = \frac{10}{2{\cdot}05} \quad \text{or} \quad \frac{55}{n} = \frac{10}{2{\cdot}05}$$

$\therefore$

$$n = 11{\cdot}275 \quad \textbf{(Ans.)}$$

(iii) Weight of spring:

Weight of spring = Volume of wire × density of the wire material

$$= \left(\frac{\pi}{4} d^2\right) \times (2\pi\, Rn) \times \text{density}$$

$$= \frac{\pi}{4} \times (11 \times 10^{-3})^2 \times (2\pi \times 55 \times 10^{-3} \times 11{\cdot}275)$$

$$\times (76{\cdot}5 \times 10^3)\ N = 28{\cdot}32 \text{ N}$$

i.e., *Weight of spring* = **28·32 N (Ans.)**

Example 14·30. *A close-coiled helical spring of 18 mm mean coil diameter and 10 turns is arranged within and concentric with an outer spring. The free length of the inner spring is 4 mm more than that of the outer. The outer spring has 12 coils of mean diameter 30 mm and wire diameter 3·5 mm. The spring load against which a valve is opened is provided by the inner spring. The initial compression in outer spring is 6 mm when the valve is closed. Calculate:*

(i) *The stiffness of the inner spring if the greatest force required to open the valve by 9 mm is 150 N.*

(ii) *The diameter of the wire of the inner spring.*

Take: $C = 80 \times 10^3$ *N/mm²*.

Solution. Let suffix '1' represent inner spring and suffix '2' represent outer spring.

Then, mean coil diameter of the *inner spring*,

$$D_1 = 18 \text{ mm}$$

Number of turns, $n_1 = 10$

Free length of inner spring = 4 mm more than the outer one

Wire diameter of *outer spring*, $d_2 = 3{\cdot}5$ mm

Mean diameter, $D_2 = 30$ mm

Number of turns, $n_2 = 12$

Initial compression = 6 mm

∴ Initial compression in the inner spring

$$= 4 + 6 = 10 \text{ mm}$$

Stiffness of inner spring:

Let, k_1 = Stiffness of inner spring in N/mm, and

k_2 = Stiffness of outer spring in N/mm.

Initial load on the valve, $F_1 = 10\,k_1 + 6\,k_2$

Stiffness of the outer spring, $k_2 = \dfrac{Cd_2^4}{64\,R_2^3\,n_2} = \dfrac{80 \times 10^3 \times 3\cdot5^4}{64 \times 15^3 \times 12} = 4\cdot63$ N/mm

The valve is to open by 9 mm, additional force required to open the valve,

$$F_2 = 9k_1 + 9k_2$$

Total load to lift the valve by 9 mm

$$F = F_1 + F_2 = 19k_1 + 15\,k_2 = 150$$

or, $19k_1 + 15 \times 4{\cdot}63 = 150$

∴ $\mathbf{k_1 = 4{\cdot}24}$ **N/mm (Ans.)**

(ii) Wire diameter of the inner spring, d_1:

$$k_1 = \frac{Cd_1^4}{64\,R_1^3\,n_1} \quad \text{or} \quad 4\cdot24 = \frac{80 \times 10^3 \times d_1^4}{64 \times 9^3 \times 10}$$

or, $d_1^4 = \dfrac{4\cdot24 \times 64 \times 9^3 \times 10}{80 \times 10^3} = 24\cdot727$

∴ $\mathbf{d_1 = 2{\cdot}23}$ **mm (Ans.)**

Example 14·31. *In a compound helical spring, the inner spring is arranged within and concentric with the outer one, but is 7 mm shorter in length. The outer spring has 12 coils of mean diameter 30 mm and the wire diameter is 3·5 mm.*

(i) Find the stiffness of the inner spring if an axial load of 150 N causes the outer spring to compress by 20 mm.

(ii) If the radial clearance between the springs is 1·5 mm, find the wire diameter of the inner spring, if it has 10 coils.

Take: $C = 77000$ *N/mm²*.

Solution. Outer Spring:

Wire diameter, $d = 3{\cdot}5$ mm

Mean coil diameter, $D = 30$ mm

Number of coils, $n = 12$

Compression, $\delta = 20$ mm

Load required, $W_{outer} = \dfrac{Cd^4\delta}{64\,R^3 n} = \dfrac{77000 \times (3\cdot5)^4 \times 20}{64 \times 15^3 \times 12} = 89\cdot16$ N

Inner Spring:

Load shared by the inner spring,

$$W_{inner} = 150 - 89{\cdot}16 = 60{\cdot}84 \text{ N}$$

Compression, $\delta = 20 - 7 = 13$ mm

Number of coils $= 10$

(i) **Stiffness of inner spring k_{inner}:**

$$k_{inner} = \frac{W_{inner}}{\delta} = \frac{60 \cdot 84}{13} = 4 \cdot 68 \text{ N/mm} \quad \text{(Ans.)}$$

(ii) **Wire diameter of the inner spring:**

Let, d = Wire diameter of the inner spring.

Now, mean coil radius of outer spring

$= 15$ mm

Wire diameter of outer spring $= 3{\cdot}5$ mm

Inner coil radius of outer spring $= 15 - \frac{3 \cdot 5}{2} = 13 \cdot 25$ mm

Radial clearance between the two springs = 1·5 mm

∴ Outer radius of inner spring $= 13{\cdot}25 - 1{\cdot}5 = 11{\cdot}75$ mm

Mean coil radius, $R_{inner} = \left(11 \cdot 75 - \frac{d}{2}\right)$ mm

Now, $$W_{inner} = \frac{77000 \times d^4 \times 13}{64 \times (11 \cdot 75 - d/2)^3 \times 10}$$

or, $$60 \cdot 84 = \frac{1564\, d^4}{(11 \cdot 75 - d/2)^3}$$

$$(11{\cdot}75 - d/2)^3 = 25{\cdot}7\, d^4$$

$$(23{\cdot}5 - d)^3 = 205{\cdot}6\, d^4$$

From which, $d = 2{\cdot}58$ mm say **2·6 mm** **(Ans.)**

Example 14·32. *A close-coiled helical spring has 30 turns, the mean radius of the coils is 75 mm while the wire diameter of the wire is 15 mm. Find the work done in rotating one end of the spring by 80° relative to the other end (fixed end), by a couple whose axis coincides with the axis of the spring.*

Take: $E = 210000 \text{ N/mm}^2$.

Another view of an automobile wheel and spring suspension.

Solution. Wire diameter of the spring,

$d = 15$ mm

Mean coil radius, $R = 75$ mm

Number of turns, $n = 30$

Young's modulus, $E = 210000 \text{ N/mm}^2$

Moment of inertia, $I = \frac{\pi}{64} d^4 = \frac{\pi}{64} \times 15^4 = 2485 \text{ mm}^4$

Length of the wire, $l = 2\pi Rn = 2\pi \times 75 \times 30 = 14137$ mm

Angular rotation, $\theta = 80° = 80 \times \pi/180 = 1{\cdot}396$ radians

We know, $$\phi = \frac{Ml}{EI}$$

∴ $$M = \frac{\phi\, EI}{l} = \frac{1 \cdot 396 \times 210000 \times 2485}{14137} \times 10^{-3} \text{ Nm} = 51 \cdot 53 \text{ Nm}$$

∴ Work done on the spring,

$$U = \frac{1}{2} M\, \theta = \frac{1}{2} \times 51 \cdot 53 \times 1 \cdot 396 = \mathbf{35 \cdot 97 \text{ Nm}} \quad \text{(Ans.)}$$

Example 14·33. *A stiff bar of negligible weight transmits a load P to combination of three springs as shown in the Fig. 14·9. The springs are made up of the same material and out of rods of equal diameters. They are of the same length before loading. The number of coils in the three springs are 10, 12 and 15 respectively, while the mean radii are in ratio of 1 : 1·2 : 1·4 respectively. Find the distance x such that the bar remains horizontal after applying the load.*

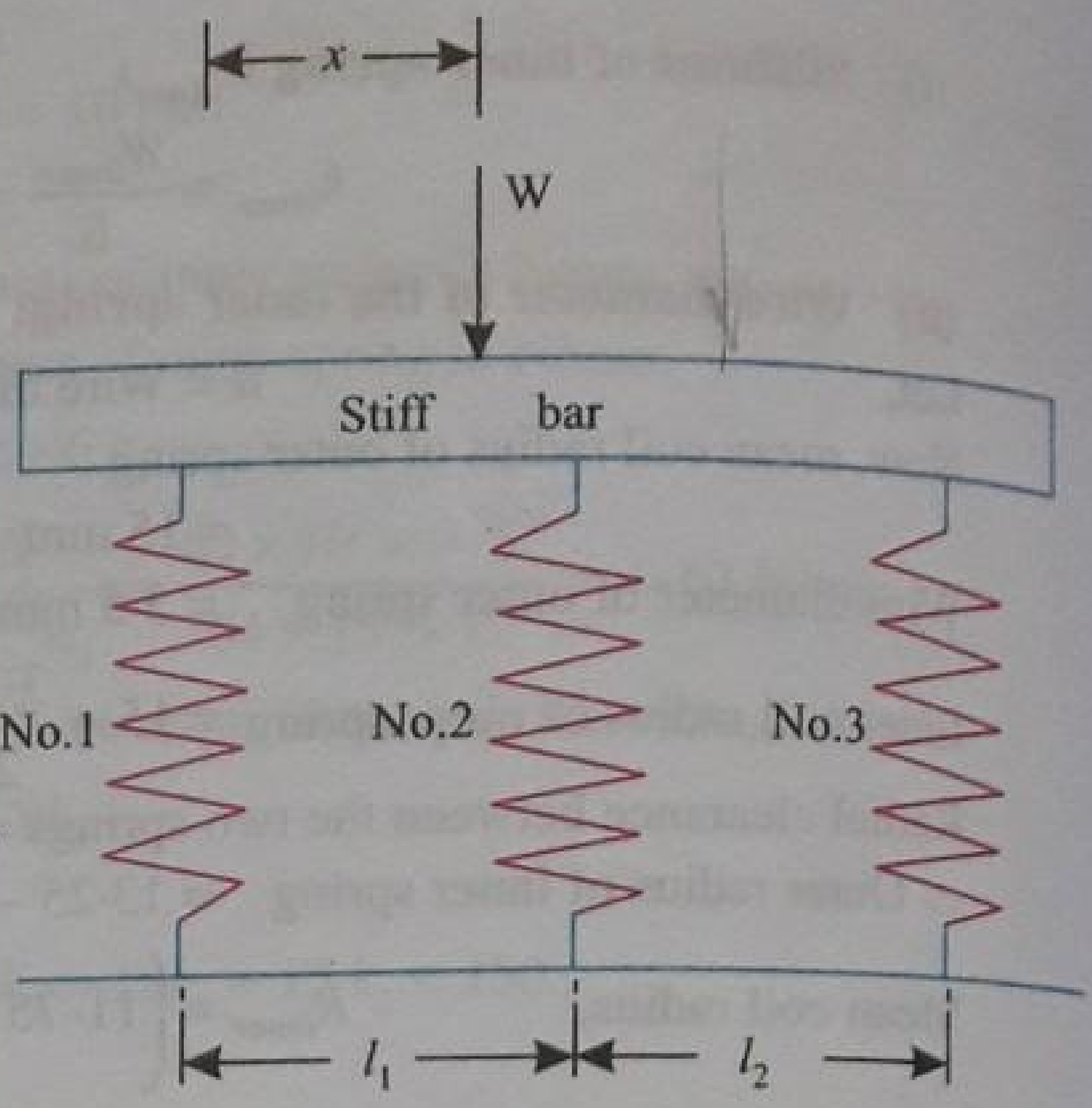


Fig. 14.9

Solution. Refer to Fig. 14·9.

The stiff bar will only remain horizontal if the springs get compressed by the same amount, say δ. Let W_1, W_2 and W_3 be the loads carried by the three springs respectively. Let the mean radii of the coils of three springs be R, $1{\cdot}2\,R$ and $1{\cdot}4\,R$ respectively as per the proportion of three coils.

For first spring:

$$\delta = \frac{64\, W_1 R_1^3\, n_1}{Cd^4} = \frac{64\, W_1 R_1^3 \times 10}{Cd^4}$$

For second spring:

$$\delta = \frac{64\, W_2 R_2^3 \times n_2}{Cd^4} = \frac{64\, W_2 R_2^3 \times 12}{Cd^4}$$

For third spring:

$$\delta = \frac{64\, W_3 R_3^3 n_3}{Cd^4} = \frac{64\, W_3 R_3^3 \times 15}{Cd^4}$$

Since δ, C and d are same for all springs and

$$R_1 = R,\ R_2 = 1{\cdot}2\,R \text{ and } R_3 = 1{\cdot}4\,R$$

$$\therefore \quad 10\,W_1 = (1{\cdot}2)^3 \times 12\,W_2 = (1{\cdot}4)^3 \times 15\,W_3$$

or,
$$W_1 = \frac{(1{\cdot}4)^3 \times 15\,W_3}{10} \quad \text{and,} \quad W_2 = \frac{(1{\cdot}4)^3 \times 15\,W_3}{(1{\cdot}2)^3 \times 12}$$

Now let the load W act at a distance x from the left end as shown. Taking moments about the point where the load is acting, we have

$$W_1\,x = W_2\,(l - x) + W_3\,(2l - x)$$

where,
$$l_1 = l_2 = l$$

or,
$$\frac{(1{\cdot}4)^3 \times 15\,W_3 \cdot x}{10} = \frac{(1{\cdot}4)^3 \times 15\,W_3}{(1{\cdot}2)^3 \times 12}(l - x) + W_3\,(2l - x)$$

$$4{\cdot}116x = 1{\cdot}985\,(l - x) + (2l - x)$$

$$4{\cdot}116x = 1{\cdot}985\,l - 1{\cdot}985x + 2l - x$$

$$4{\cdot}116x = 3{\cdot}985\,l - 2{\cdot}985x$$

$$7{\cdot}101x = 3{\cdot}985l$$

$$x = \mathbf{0{\cdot}561}\boldsymbol{l} \quad \textbf{(Ans.)}$$

Example 14·34. *Find the weight of a close-coiled helical spring which would absorb the energy of truck weighing 95 kN and moving with a velocity of 1·2 m / s if:*

(*i*) *spring is compressed by the impact,*

(*ii*) *spring is wound up by the impact.*

Working stress: 290 MN/m² (bending); 240 MN/m² (torsion)

$E = 200$ GN/m², $C = 80$ GN/m²; specific gravity of material = 7·9. **(Banglore University)**

Solution. Kinetic energy to be used,

$$K.E. = \frac{1}{2}\frac{W}{g}v^2$$

where, W = Weight of truck = 95 kN,

v = Velocity of truck = 1·2 m/s, and

g = Acceleration due to gravity = 9·81 m/s².

i.e.,

$$K.E. = \frac{1}{2} \times \frac{95}{9 \cdot 81} \times 1 \cdot 2^2 = 6 \cdot 97 \text{ kNm}$$

(*i*) Spring is compressed by impact:

If the spring absorbs the energy in direct compression, then the wire of the spring is in torsion. Hence strain energy,

$$U = \frac{\tau^2}{4C} \times \text{volume of spring}$$

Equating the strain energy to *K.E.* and substituting values, we get

$$\frac{\tau^2}{4C} \times \text{volume of spring} = K.E.$$

$\therefore$ Volume of spring $= \dfrac{K.E. \times 4\,C}{\tau^2} = \dfrac{(6 \cdot 97 \times 10^3) \times 4 \times 80 \times 10^9}{(240 \times 10^6)^2} = 0 \cdot 03872 \text{ m}^3$

Weight of spring = Volume of spring × specific gravity × density of water

= 0·03872 × 7·9 × 9·81 kN ($\because$ Density of water = 9·81 kN/m³)

= **3 kN (Ans.)**

Drawbars behind a tractor are meant to pull the loads.

(ii) Spring is wound up by impact:

Strain energy, $U = \dfrac{\sigma_b^2}{8E} \times$ volume (where, σ_b = maximum bending stress in the wire)

Equating strain energy to *K.E.*, we get

$$\frac{\sigma_b^2}{8E} \times \text{volume of spring} = K.E.$$

$$\text{Volume of spring} = \frac{K.E. \times 8E}{\sigma_b^2} = \frac{6.97 \times 10^3 \times 8 \times 200 \times 10^9}{(290 \times 10^6)^2} = 0.1326 \text{ m}^3$$

and, weight of spring = Volume of spring × density

$= 0.1326 \times (7.9 \times 9.81)$ kN

$=$ **10·276 kN (Ans.)**

HIGHLIGHTS

1. Types of springs: (*i*) Helical springs, (*ii*) Leaf springs, (*iii*) Torsion springs, (*iv*) Circular springs, (*v*) Belleville springs and (*vi*) Flat springs.
2. *Close-coiled helical springs* subjected to '*axial load*':

Shear stress, $\tau = \dfrac{16\,WR}{\pi d^3}$ ...(*i*)

Angle of twist, $\theta = \dfrac{64\,WR^2 n}{Cd^4}$ ...(*ii*)

Deflection, $\delta = R\theta$ ...(*iii*)

or, $\delta = \dfrac{64\,WR^3 n}{Cd^4}$ ...(*iv*)

Stiffness, $k = \dfrac{W}{\delta} = \dfrac{Cd^4}{64\,R^3 n}$ ...(*v*)

Energy stored, $U = \dfrac{1}{2} \times T \times \theta$ ...(*vi*)

$U = \dfrac{\tau^2}{4C} \times$ volume of spring ...(*vii*)

$U = \dfrac{1}{2} W\delta$

3. Helical springs subjected to '*axial twist*':

Angle of twist, $\theta = \dfrac{Ml}{EI}$ ...(*i*)

$\theta = \dfrac{128\,MRn}{Ed^4}$ ...(*ii*)

Bending stress, $\sigma_b = \dfrac{32M}{\pi d^3}$ ...(*iii*)

Energy stored, $U = \dfrac{\sigma_b^2}{8E} \times$ volume of spring wire ...(*iv*)

4. *Open-coiled helical spring*

(i) *With axial load:*

The couple producing *torsion*,

$$T = W R \cos \alpha$$

The couple producing *bending*,

$$M = W R \sin \alpha$$

Deflection, $$\delta = 2 W R^3 n \pi \sec \alpha \left[\frac{\cos^2 \alpha}{C I_p} + \frac{\sin^2 \alpha}{E I} \right]$$

Angular rotation, $$\psi = 2 W R^2 n \pi \sin \alpha \left[\frac{1}{C I_p} - \frac{1}{E I} \right]$$

where, W = Applied load, R = Mean coil radius,
n = Number of coils, α = Helix angle,
C = Modulus of rigidity, and E = Young's modulus.

(ii) *With axial thrust:*

Angular rotation, $$\phi = 2 T R n \pi \sec \alpha \left[\frac{\sin^2 \alpha}{C I_p} + \frac{\cos^2 \alpha}{E I} \right]$$

Deflection, $$\delta = 2 T R^2 n \pi \sin \alpha \left[\frac{1}{C I_p} - \frac{1}{E I} \right]$$ (where, T = Applied torque)

5. *Springs in series:*

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

Springs in parallel: $k = k_1 + k_2$

where, k_1 = Stiffness of spring 1,
k_2 = Stiffness of spring 2, and
k = Stiffness of composite spring.

6. *Flat spiral spring:*

Angular rotation, $$\phi = \frac{Tl}{EI} \quad \text{...(i)}$$

Maximum bending moment,

$$M_{max} = 2T \quad \text{...(ii)}$$

Maximum stress, $$\sigma_{max} = \frac{12T}{bt^2} \quad \text{...(iii)}$$

Winding torque, $$T = \frac{bt^2 \cdot \sigma_{max}}{12} \quad \text{...(iv)}$$

Energy stored, $$U = \frac{6T^2 l}{Ebt^3} \quad \text{...(v)}$$

Maximum energy stored,

$$U_{max} = \frac{\sigma_{max}^2}{24 E} \times \text{volume of spring} \quad \text{...(vi)}$$

Resilience of spring (= energy per unit volume) $= \dfrac{\sigma_{max}^2}{24E}$ ...(vii)

7. *Laminated spring:*

A. *Semi-elliptical spring:*

Bending stress, $\sigma_b = \dfrac{3\,Wl}{2\,Nbt^2}$ ...(i)

Deflection, $\delta = \dfrac{3\,Wl^3}{8\,EN\,bt^3}$ ...(ii)

Strain energy, $U = \dfrac{\sigma_b^2}{6E} \times$ volume of spring ...(iii)

Where, W = Applied load,
b = Width of each plate,
t = Thickness of each plate,
N = Number of plates in the spring,
l = The spring span length, and
E = Young' modulus of elasticity.

B. *Quarter elliptical spring:*

Bending stress, $\sigma_b = \dfrac{6\,Wl}{Nbt^2}$ ...(i)

Deflection, $\delta = \dfrac{6\,Wl^3}{ENbt^3}$ ...(ii)

OBJECTIVE TYPE QUESTIONS

Choose the Correct Answer:

1. If a close-coiled helical spring is subjected to load W and the deflection produced is δ, then stiffness of the spring is given by
   (a) W/δ (b) $W\cdot\delta$
   (c) δ/W (d) $W^2\cdot\delta$.

2. Wahl's connection factor (K) is given by the relation
   (a) $K = \dfrac{3S-1}{3S-4} + \dfrac{0\cdot615}{S}$
   (b) $K = \dfrac{4S-1}{4S-4} + \dfrac{0\cdot615}{S}$
   (c) $K = \dfrac{5S-1}{5S-4} + \dfrac{0\cdot615}{S}$
   (d) $K = \dfrac{6S-1}{6S-4} + \dfrac{0\cdot615}{S}$.
   (where S = spring index)

3. The energy stored in a close-coiled helical spring, when subjected to an 'axial twist', is given by
   (a) $\dfrac{\sigma_b^2}{6E} \times$ volume of spring
   (b) $\dfrac{\sigma_b^2}{8E} \times$ volume of spring
   (c) $\dfrac{\sigma_b^2}{4E} \times$ volume of spring
   (d) $\dfrac{\sigma_b^2}{2E} \times$ volume of spring.
   (where s_b = bending stress)

4. Two springs of stiffness k_1 and k_2 respectively are connected in series, the stiffness of the composite spring (k) will be given by
   (a) $k = k_1 + k_2$ (b) $k = k_1 \times k_2$
   (c) $k = \dfrac{k_1 k_2}{k_1 + k_2}$ (d) $k = \dfrac{k_1 + k_2}{k_1 k_2}$.

5. The resilience of a flat spiral spring is given by

(a) $\frac{\sigma_{max}}{24E}$ (b) $\frac{\sigma^2_{max}}{24E}$

(c) $\frac{\sigma^2_{max}}{12E}$ (d) $\frac{\sigma^2_{max}}{8E}$.

(where, σ_b = bending stress)

6. In case of a laminated spring, the load at which the plates become straight is called
   (a) working load (b) safe load
   (c) proof load (c) none of the above.
7. .... are called cantilever laminated springs.
   (a) Semi-elliptical springs
   (b) Quarter elliptical springs
   (c) Both (a) and (b)
   (d) None of the above.

Even good seats need springs to cushion shocks. The above picture shows a tractor driver seat.

ANSWERS

1. (a)	2. (b)	3. (b)	4. (c)	5. (b)	6. (c)	7. (b)

UNSOLVED EXAMPLES

Close-coiled helical springs :

1. A close-coiled helical spring, with the coil diameter as 100 mm and wire diameter as 12 mm consists of 16 coils. If it is subjected to an axial tension of 400 N, find the maximum stress induced in the coil, the extension suffered by the spring and the energy stored in it. Modulus of rigidity, C = 84 GN/m^2. [Ans. 59 MN/m^2; 29·4 mm; 5·88 Nm]
2. A close-coiled helical spring having 100 mm mean diameter is made of 20 turns of 10 mm diameter steel rod. The spring carries an axial load of 100 N. Find the shearing stress developed in the spring and the deflection of the load. C = 84 GN/m^2.
   [Ans. 25·5 MN/m^2; 19·1 mm]
3. A close-coiled helical spring, made of 6 mm diameter steel wire has 20 coils, each of 100 mm mean diameter. When subjected to axial load of 70 N, calculate:
   (i) The maximum shear stress produced, (ii) The deflection,
   (iii) Stiffness, and (iv) The energy stored.
   Take: C = 84 GN/m^2. [Ans. 82·6 MN/m^2; 103 mm; 0·68 N/mm; 3·6 Nm]
4. A close-coiled helical spring is to carry a load of 100 N and the mean coil diameter is to be eight times the wire diameter. Calculate these diameters if the maximum shear stress is to be 75 MN/m^2.
   [Ans. 5·21 mm; 41·68 mm]
5. A close-coiled spring has a radius of 40 mm and length 320 mm. It is required to extend 21 mm under a pull of 185 N. If C = 84 GN/m^2 determine the diameter of the wire.
   [Ans. 4·84 mm]
6. The mean diameter of a spring is six times the diameter of the wire. It extends by 60 mm under a pull of 550 N. If the maximum allowable shear stress is 350 MN/m^2, find the size of the wire and the number of coils. Take C = 84·4 MN/m^2. [Ans. d = 4·9 mm; n = 26·1]