SUBJECT: SIGNALS & SYSTEMS

PART—A

UNIT 1:

Introduction: Definitions of a signal and a system, classification of signals, basic Operations on signals, elementary signals, Systems viewed as Interconnections of operations, properties of systems.

07 Hours

UNIT 2:

Time-domain representations for LTI systems – 1: Convolution, impulse response representation, Convolution Sum and Convolution Integral.

06 Hours

UNIT 3:

Time-domain representations for LTI systems — 2: properties of impulse response representation, Differential and difference equation Representations, Block diagram representations.

07 Hours

UNIT 4:

Fourier representation for signals — 1: Introduction, Discrete time and continuous time Fourier series (derivation of series excluded) and their properties

06 Hours

PART—B

UNIT 5:

Fourier representation for signals — 2: Discrete and continuous Fourier transforms(derivations of transforms are excluded) and their properties.

06 Hours

UNIT 6:

Applications of Fourier representations: Introduction, Frequency response of LTI systems, Fourier transform representation of periodic signals, Fourier transform representation of discrete time signals.

07 Hours

UNIT 7:

Z-Transforms — 1: Introduction, Z — transform, properties of ROC, properties of Z — transforms, inversion of Z — transforms.

07 Hours

UNIT 8:

Z-transforms — 2: Transform analysis of LTI Systems, unilateral Z Transform and its application to solve difference equations. **06 Hours**

INDEX

UNIT 1: Introduction **Teaching hours: 7**

Introduction: Definitions of a signal and a system, classification of signals, basic Operations on signals, elementary signals, Systems viewed as Interconnections of operations, properties of systems.

Unit 1:
Introduction

1.1.1 Signal definition

A signal is a function representing a physical quantity or variable, and typically it contains information about the behaviour or nature of the phenomenon.

For instance, in a RC circuit the signal may represent the voltage across the capacitor or the current flowing in the resistor. Mathematically, a signal is represented as a function of an independent variable 't'. Usually 't' represents time. Thus, a signal is denoted by x(t).

1.1.2 System definition

A system is a mathematical model of a physical process that relates the input (or excitation) signal to the output (or response) signal.

Let x and y be the input and output signals, respectively, of a system. Then the system is viewed as a transformation (or mapping) of x into y. This transformation is represented by the mathematical notation

31=Tx (1.1)

where **T** is the operator representing some well-defined rule by which x is transformed into y. Relationship (1.1) is depicted as shown in Fig. 1-1(a). Multiple input and/or output signals are possible as shown in Fig. 1-1(b). We will restrict our attention for the most part in this text to the single-input, single-output case.

1.1 System with single or multiple input and output signals

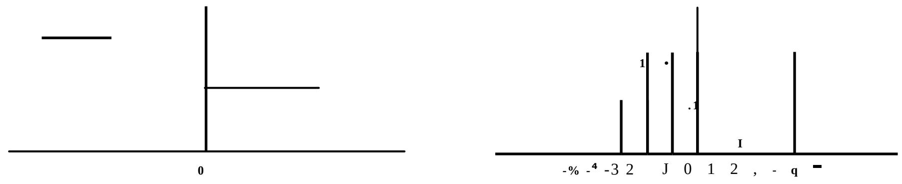
1.2 Classification of signals

Basically seven different classifications are there:

4- Continuous-Time and Discrete-Time Signals
A. Analog and Digital Signals
Real and Complex Signals
A. Deterministic and Random Signals
4- Even and Odd Signals
4- Periodic and Nonperiodic Signals
4- Energy and Power Signals

Continuous-Time and Discrete-Time Signals

A signal x(t) is a continuous-time signal if t is a continuous variable. If t is a discrete variable, that is, x(t) is defined at discrete times, then x(t) is a discrete-time signal. Since a discrete-time signal is defined at discrete times, a discrete-time signal is often identified as a sequence of numbers, denoted by {x,) or x[n], where n = integer. Illustrations of a continuous-time signal x(t) and of a discrete-time signal x[n] are shown in Fig. 1-2.



1.2 Graphical representation of (a) continuous-time and (b) discrete-time signals

Analog and Digital Signals

If a continuous-time signal x(t) can take on any value in the continuous interval (a, b), where a may be - 00 and b may be +00 then the continuous-time signal x(t) is called an analog signal. If a discrete-time signal x[n] can take on only a finite number of distinct values, then we call this signal a digital signal.

Real and Complex Signals

A signal x(t) is a real signal if its value is a real number, and a signal x(t) is a complex signal if its value is a complex number. A general complex signal x(t) is a function of the form

$$x(t) = x_i(t) + jx2(t) \quad \text{———} \quad 12$$

where xi (t) and x2 (t) are real signals and j = V-1

Note that in Eq. (1.2) **'t'** represents either a continuous or a discrete variable.

Deterministic and Random Signals:

Deterministic signals are those signals whose values are completely specified for any given time. Thus, a deterministic signal can be modelled by a known function of time **'t'.**

Random signals are those signals that take random values at any given time and must be characterized statistically.

Even and Odd Signals

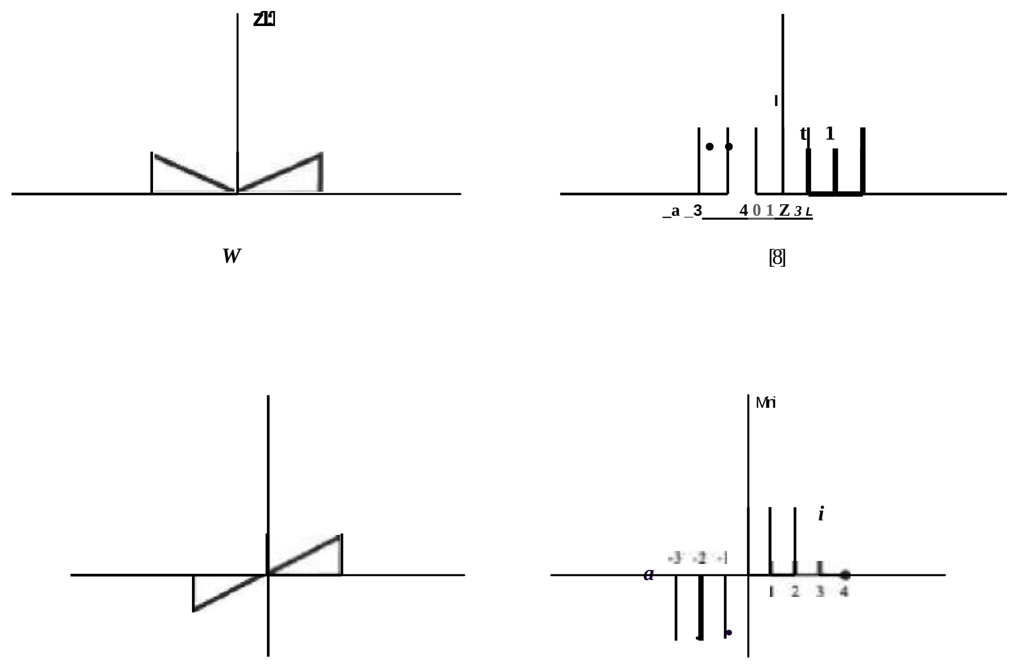
A signal x ***(t)*** or ***x[n]*** is referred to as an ***even*** signal if

$$x(-t) = x(t)$$
$$x[-n] = x[n] \quad \text{———} \quad (1.3)$$

A signal x ***(t)*** or ***x[n]*** is referred to as an ***odd*** signal if

$$x(-t) = -x(t)$$
$$x[-n] \quad -x[n] \quad \text{———} \quad (1.4)$$

Examples of even and odd signals are shown in Fig. 1.3.



1.3 Examples of even signals (a and b) and odd signals (c and d).

Any signal x(t) or x[n] can be expressed as a sum of two signals, one of which is even and one of which is odd. That is,

$$X(t) \equiv X_o(t) \pm Xe\,Lt \qquad (1.5)$$

Where,

$$;(t) = \frac{1}{2}(x(0 + x(-0)$$

$$x_o(t) = \frac{1}{,}(x(t) - x(-0) \qquad (1.6)$$

Similarly for x[n],

$$X[n] = {}_o[n] + x_e[n] \qquad (1.7)$$

Where,

$$X_e[n] = \frac{1}{2}(x[n] + x[-n])$$

$${}_o[n] = \frac{1}{-}\,1X[0] - x[-u]) \qquad (1.8)$$

Note that the product of two even signals or of two odd signals is an even signal and that the product of an even signal and an odd signal is an odd signal.

Periodic and Nonperiodic Signals

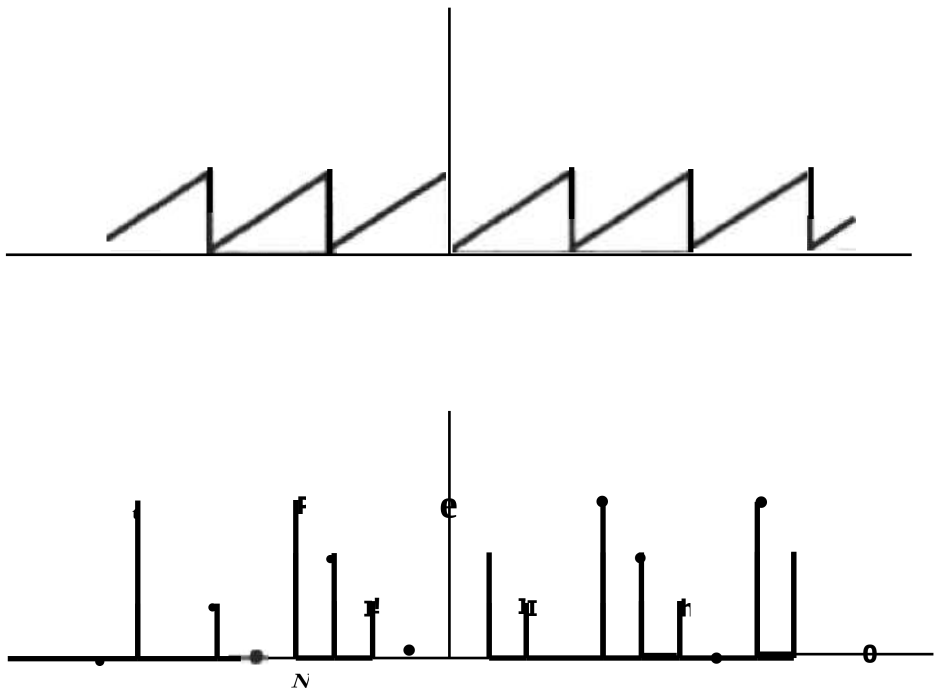
A continuous-time signal x (t) is said to be periodic with period T if there is a positive nonzero value of T for which

$$+ \quad - x(e) \qquad all\,d \qquad (1.9)$$

An example of such a signal is given in Fig. 1-4(a). From Eq. (1.9) or Fig. 1-4(a) it follows that

xi r + ***en T)*** (1.10)

for all t and any integer m. The fundamental period T, of x(t) is the smallest positive value of T for which Eq. (1.9) holds. Note that this definition does not work for a constant



i b)

1.4 Examples of periodic signals.

signal x(t) (known as a dc signal). For a constant signal x(t) the fundamental period is undefined since x(t) is periodic for any choice of T (and so there is no smallest positive value). Any continuous-time signal which is not periodic is called a nonperiodic (or aperiodic) signal.

Periodic discrete-time signals are defined analogously. A sequence (discrete-time signal) x[n] is periodic with period N if there is a positive integer N for which

n N] = x [n] (1.11)

An example of such a sequence is given in Fig. 1-4(b). From Eq. (1.11) and Fig. 1-4(b) it follows that

n + mNi — n (1.12)

for all n and any integer m. The fundamental period N_o of x[n] is the smallest positive integer N for which Eq.(1.11) holds. Any sequence which is not periodic is called a nonperiodic (or aperiodic sequence.

Note that a sequence obtained by uniform sampling of a periodic continuous-time signal may not be periodic. Note also that the sum of two continuous-time periodic signals may not be periodic but that the sum of two periodic sequences is always periodic.

Energy and Power Signals

Consider v(t) to be the voltage across a resistor R producing a current i(t). The instantaneous power p(t) per ohm is defined as

$$P(r) = \frac{v(t)i(t)}{R} = \qquad (1.13)$$

Total energy E and average power P on a per-ohm basis are

$$E \quad f_{-} i^2(i)6(1 \text{ joules}$$

$$P \quad \lim_{T1,} \frac{1}{\;} f_{-T/1}^{T/2} i^2(r)di \quad \text{watts} \qquad (1.14)$$

For an arbitrary continuous-time signal x(t), the normalized energy content E of x(t) is defined as

$$E \quad \int_{-\infty} | x(i) '\, de \qquad (1.15)$$

The normalized average power P of x(t) is defined as

$$- \text{uirn}_{TIC} \; \frac{1}{/} \; {}_{Ti\,2}^{T/2} Ix(i)r^2 di \qquad (1.16)$$

Similarly, for a discrete-time signal x[n], the normalized energy content *E* of x[n] is defined as

$$- \quad V \quad x | n \qquad (1.17)$$

The normalized average power P of x[n] is defined as

$$P = lint \; \frac{}{2N\; 4- 1 - -} \quad x | n \qquad (1.18)$$

Based on definitions (1.15) to (1.18), the following classes of signals are defined:

1. x(t) (or x[n]) is said to be an energy signal (or sequence) if and only if 0 < E < m, and so P = 0.
2. x(t) (or x[n]) is said to be a power signal (or sequence) if and only if 0 < P < m, thus implying that E = m.
3. Signals that satisfy neither property are referred to as neither energy signals nor power signals.

Note that a periodic signal is a power signal if its energy content per period is finite, and then the average power of this signal need only be calculated over a period

1.3 Basic Operations on signals

The operations performed on signals can be broadly classified into two kinds

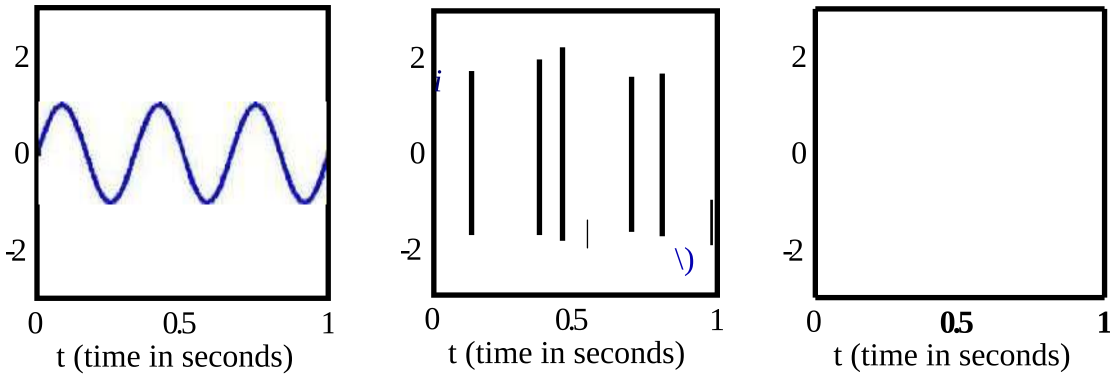
Operations on dependent variables
A- Operations on independent variables

Operations on dependent variables

The operations of the dependent variable can be classified into five types: amplitude scaling, addition, multiplication, integration and differentiation.

Amplitude scaling
Amplitude scaling of a signal *x(t)* given by equation 1.19, results in amplification of *x(t)* if *a >1*, and attenuation if *a <1*.

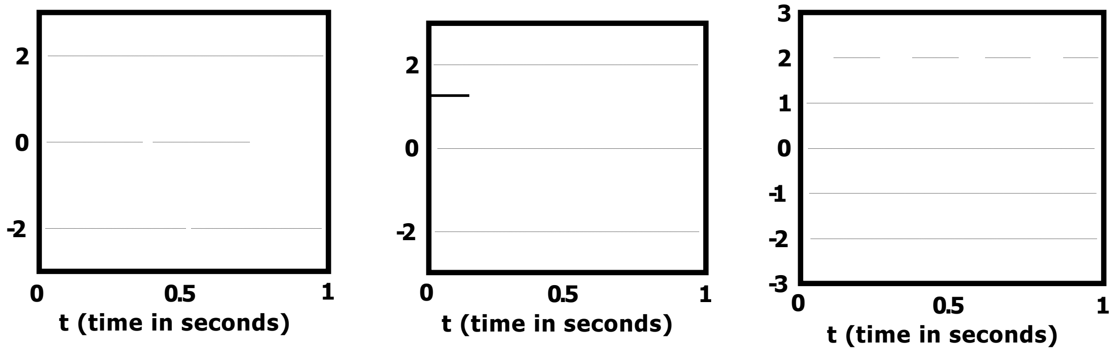
$$y(t) = ax(t) \qquad (1.20)$$



1.5 Amplitude scaling of sinusoidal signal

Addition
The addition of signals is given by equation of 1.21.

$$y(t) = x1(t) + x2(t) \qquad (1.21)$$



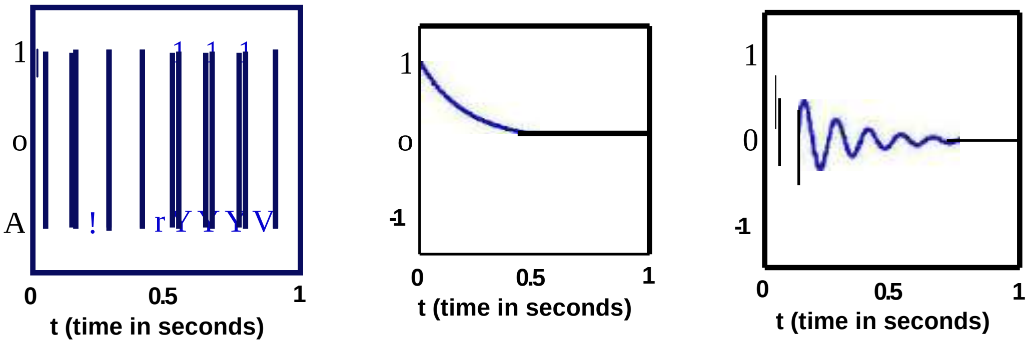
1.6 Example of the addition of a sinusoidal signal with a signal of constant amplitude (positive constant)

Physical significance of this operation is to add two signals like in the addition of the background music along with the human audio. Another example is the undesired addition of noise along with the desired audio signals.

Multiplication

The multiplication of signals is given by the simple equation of 1.22.

$$y(t) = \mathrm{xl}(t).\mathrm{x2}\,(t) \qquad (1.22)$$



1.7 Example of multiplication of two signals

Differentiation

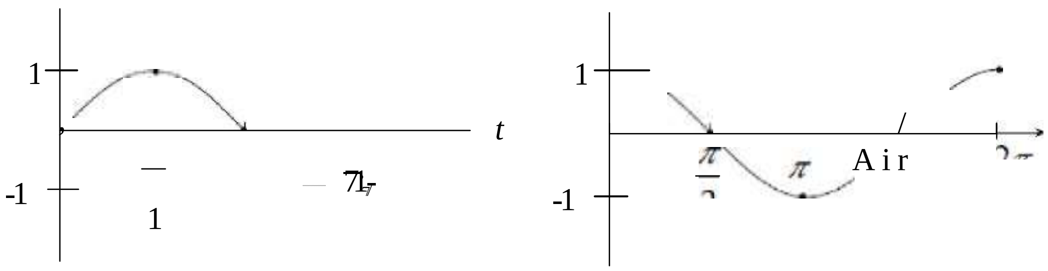
The differentiation of signals is given by the equation of 1.23 for the continuous.

$$y(t) = \frac{d}{dt}x(1) \qquad 1.23$$

The operation of differentiation gives the rate at which the signal changes with respect to time, and can be computed using the following equation, with At being a small interval of time.

$$\frac{rl}{ral}x(t) = \text{'yo-} \frac{x(t + At) - x(t)}{At} \qquad 1.24$$

If a signal doesn't change with time, its derivative is zero, and if it changes at a fixed rate with time, its derivative is constant. This is evident by the example given in figure 1.8.

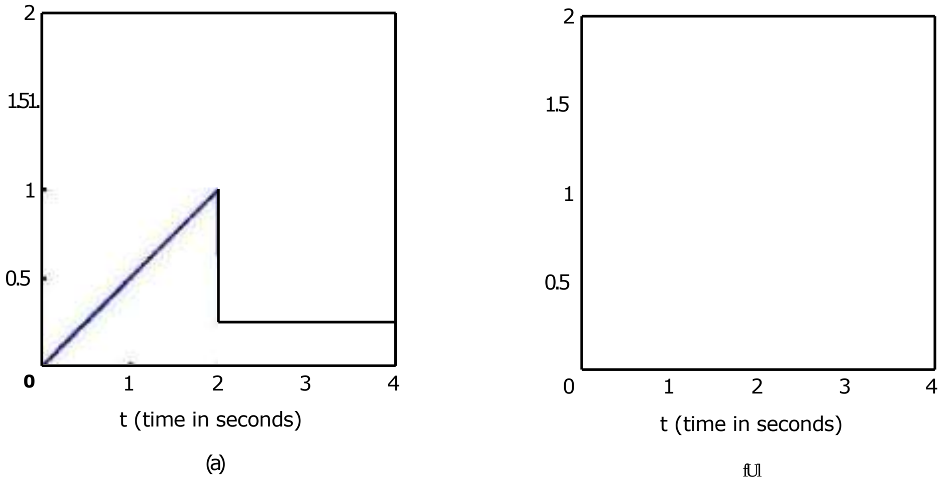


1.8 Differentiation of Sine - Cosine

Integration

The integration of a signal *x(t)* , is given by equation 1.25

$$y(t) \quad \int^{r}_{J} (\ r\ )ti\ r \qquad 1.25$$



1.9 Integration of x(t)

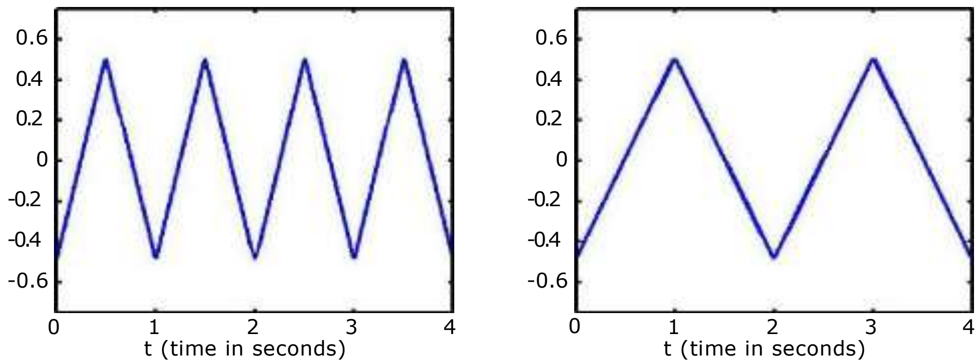
Operations on independent variables

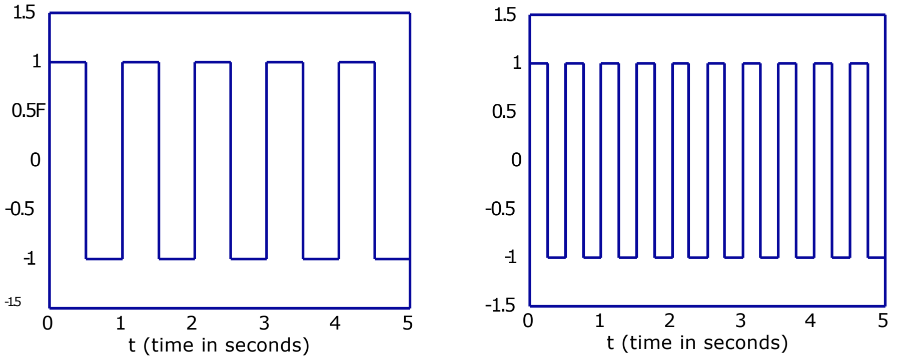
Time scaling

Time scaling operation is given by equation 1.26

$$y(t) = x(at) \qquad 1.26$$

This operation results in expansion in time for a<1 and compression in time for a>1, as evident from the examples of figure 1.10.





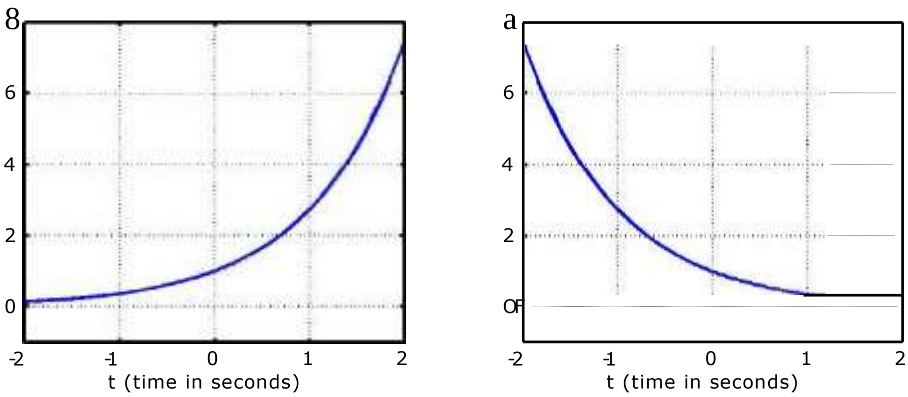
1.10 Examples of time scaling of a continuous time signal

An example of this operation is the compression or expansion of the time scale that results in the *fast-forward'* or the *'slow motion'* in a video, provided we have the entire video in some stored form.

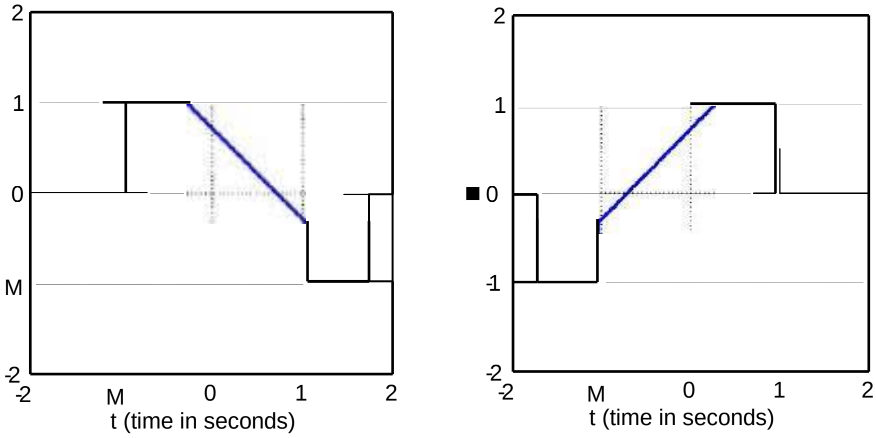
Time reflection

Time reflection is given by equation (1.27), and some examples are contained in fig1.11.

$$y(t) = x(-t) \qquad 1.27$$



(a)



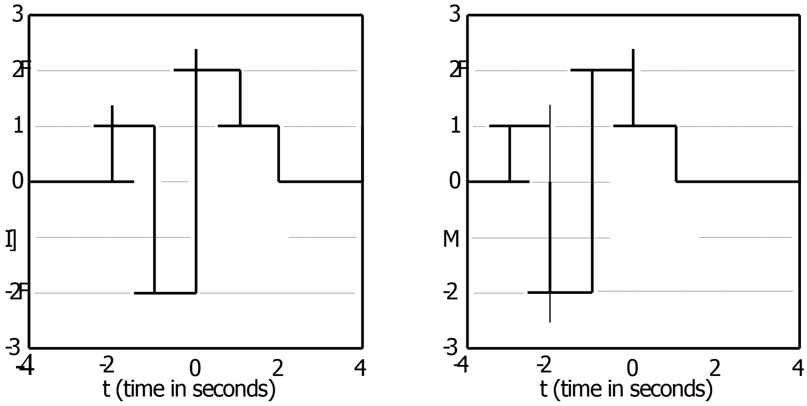
(b)

1.11 Examples of time reflection of a continuous time signal

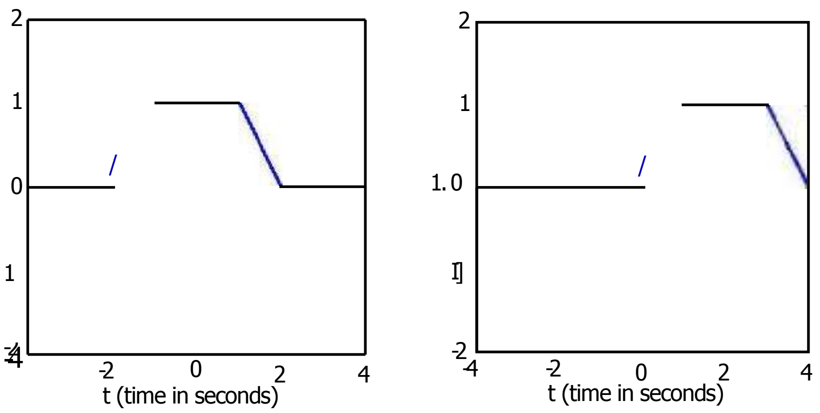
Time shifting

The equation representing time shifting is given by equation (1.28), and examples of this operation are given in figure 1.12.

$$y(t) = x(t - t0) \qquad 1.28$$



(a)



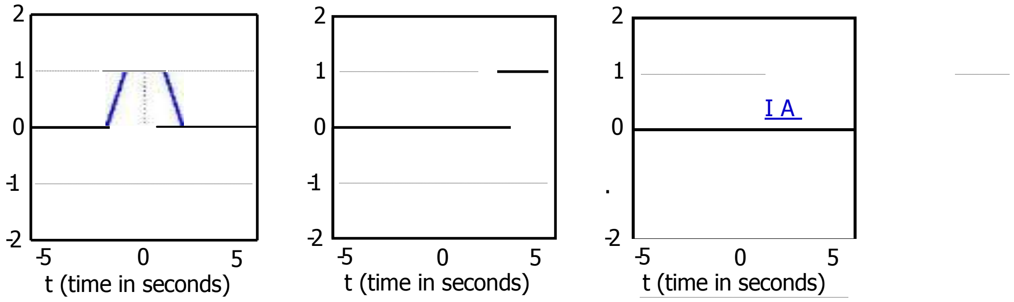
(b)

1.12 Examples of time shift of a continuous time signal

Time shifting and scaling

The combined transformation of shifting and scaling is contained in equation (1.29), along with examples in figure 1.13. Here, time shift has a higher precedence than time scale.

$$y(t) = x(at — t0) \qquad 1.29$$



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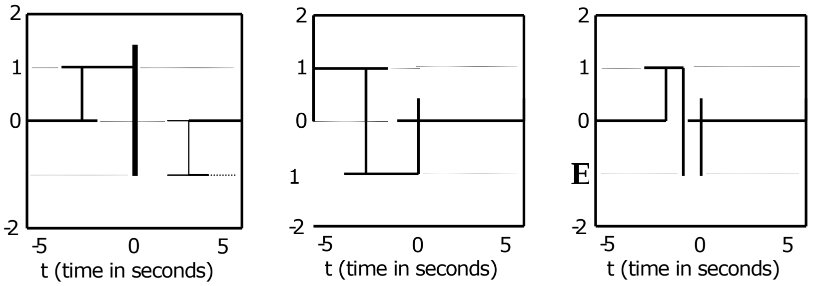


(b)

1.13 Examples of simultaneous time shifting and scaling. The signal has to be shifted first and then time scaled.

1.4 Elementary signals

Exponential signals:

The exponential signal given by equation (1.29), is a monotonically increasing function if $a > 0$, and is a decreasing function if $a < 0$.

$$x(1) = c^{Ul} \qquad \text{..............................} \textbf{(1.29)}$$

It can be seen that, for an exponential signal,

$$x(t + a^{-1}) = e.x(t)$$

$$x(1 — a^{-1}) \quad {}^{-1}.x(t) \qquad \text{.............................}\textbf{(1.30)}$$

Hence, equation (1.30), shows that change in time by ±1/ a seconds, results in change in magnitude by e±1 . The term 1/ a having units of time, is known as the time-constant. Let us consider a decaying exponential signal

$$x(!) = e^{-at} \qquad \text{for } t > 0 \qquad \text{..................} \textbf{(1.31)}$$

This signal has an initial value x(0) =1, and a final value x(oo) 0 . The magnitude of this signal at five times the time constant is,

$$x(51\ a) \quad 6.7\text{x}10^{-3} \qquad \text{............................}\textbf{(1.32)}$$

while at ten times the time constant, it is as low as,

$$x(10\ /\ a) \quad 4.5\text{x}10^{-;} \qquad \text{..................} \textbf{(1.33)}$$

-

It can be seen that the value at ten times the time constant is almost zero, the final value of the signal. Hence, in most engineering applications, the exponential signal can be said to have reached its fmal value in about ten times the time constant. If the time constant is 1 second, then final value is achieved in 10 seconds!! We have some examples of the exponential signal in figure 1.14.

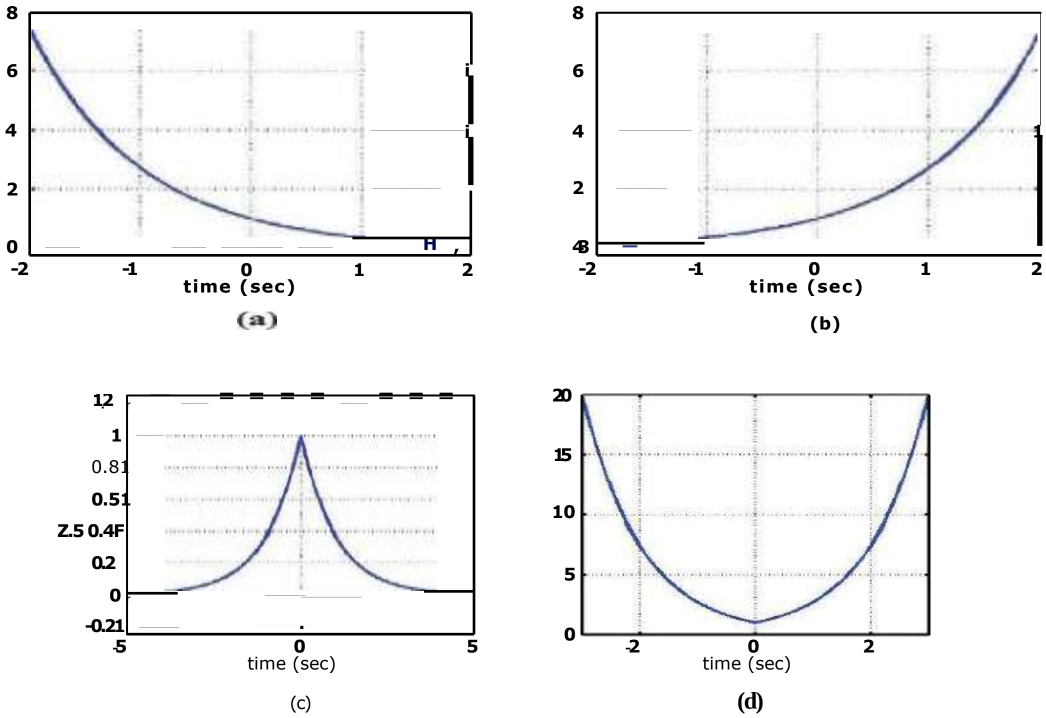


Fig 1.14 The continuous time exponential signal (a) *e—t* , (b) *et* , (c) *e—Iti* , and (d) *elti*

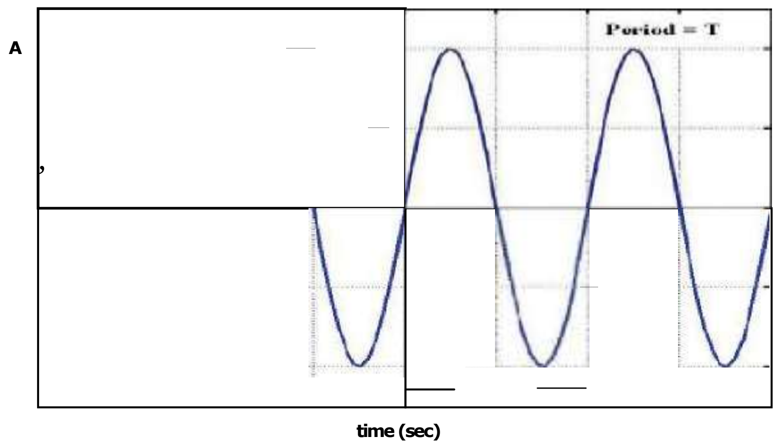
The sinusoidal signal:

The sinusoidal continuous time periodic signal is given by equation 1.34, and examples are given in figure 1.15

$$x(t) = A \sin(br\ ft) \qquad (1.34)$$

The different parameters are:

Angular frequency w = 2nf in radians,
Frequencyfin Hertz, (cycles per second)
Amplitude *A* in Volts (or Amperes)
Period *T*in seconds



The complex exponential:

We now represent the complex exponential using the Euler's identity (equation (1.35)),

$$e \quad = (\cos + j \sin 0)$$

..................(1.35)

to represent sinusoidal signals. We have the complex exponential signal given by equation (1.36)

$$e^{j\omega t} = (\cos(\omega t) + j\sin(\omega t))$$

$$e^{-j\omega t} = (\cos(\omega t) - j\sin(\omega t)) \quad \text{..........(1.36)}$$

Since sine and cosine signals are periodic, the complex exponential is also periodic with the same period as sine or cosine. From equation (1.36), we can see that the real periodic sinusoidal signals can be expressed as:

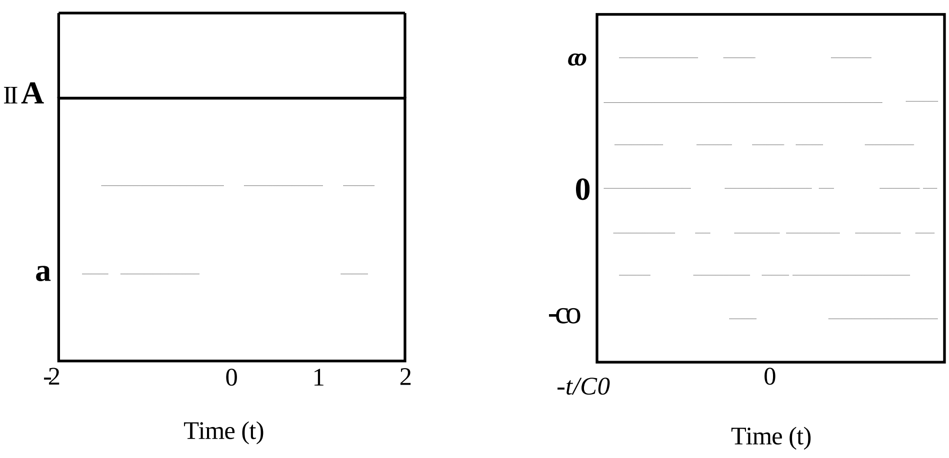
$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \quad \text{....................(1.37)}$$

Let us consider the signal *x(t)* given by equation (1.38). The sketch of this is given in fig 1.15

$$x(t) = A(t)e^{j\theta(t)} \quad \text{................................(1.38)}$$

x(t) $Ae^{j\omega t}$

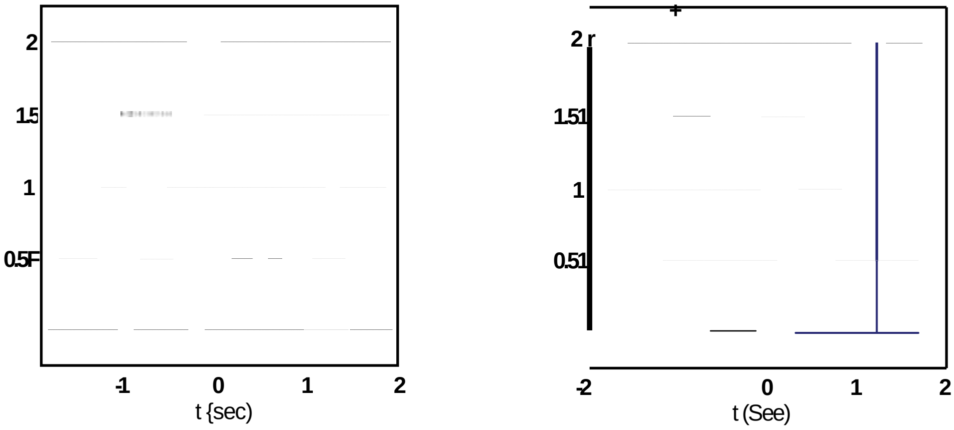


The unit impulse:

The unit impulse usually represented as $\delta(t)$, also known as the dirac delta function, is given by,

$$\delta(t) = 0 \text{ for } t \neq 0 \text{ and } \int \delta(\tau)d\tau = 1 \quad \text{....... (1.38)}$$

From equation (1.38), it can be seen that the impulse exists only at $t = 0$, such that its area is 1. This is a function which cannot be practically generated. Figure 1.16, has the plot of the impulse function



The unit step:

The unit step function, usually represented as *u(t)* , is given by,

$$H(t) = \begin{cases} 1 & t \ \text{O} \\ \text{C} \ / & t \ \ 0 \end{cases}$$

....................(1.39)

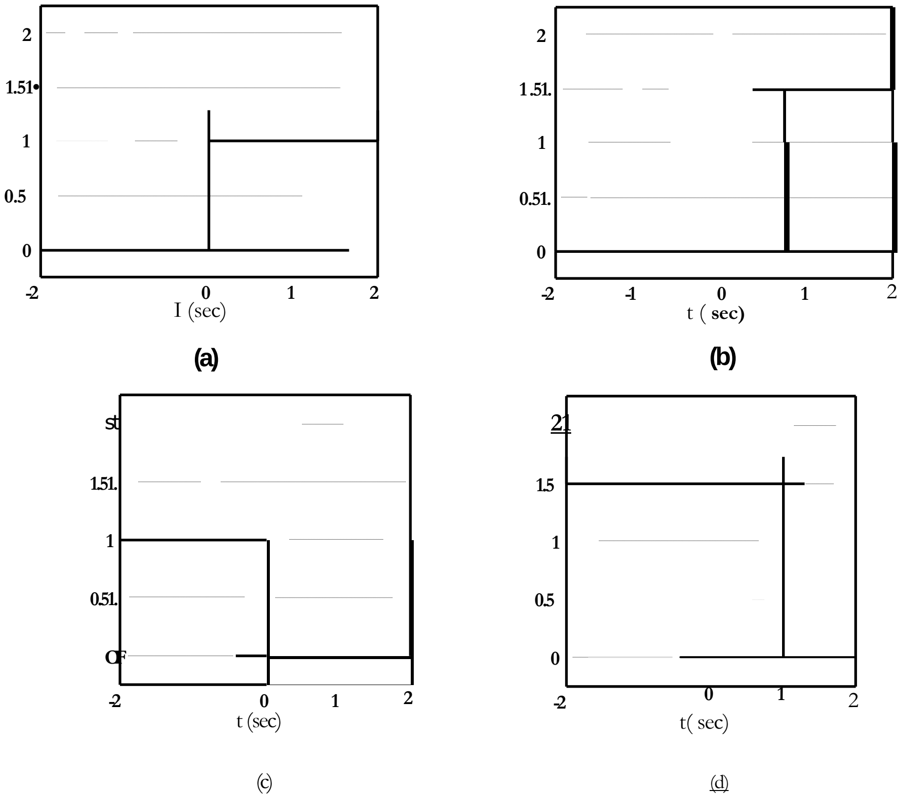


Fig 1.17 Plot of the unit step function along with a few of its transformations

The unit ramp:

The unit ramp function, usually represented as *r(t) ,* is given by,

r *r* *(j)*

7{r'1

r . 0 **................ (1.40)**

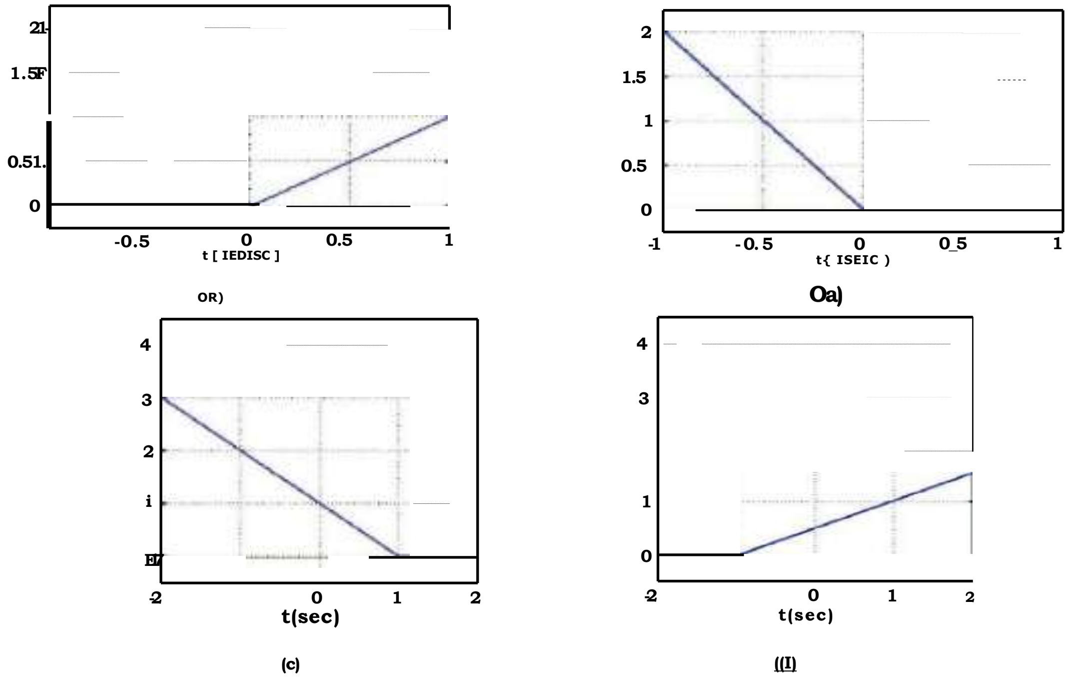


Fig 1.18 Plot of the unit ramp function along with a few of its transformations

The signum function:

The signum function, usually represented as sgn(t) , is given by

$$\mathrm{Sg11}(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < \mathbf{0} \end{cases} \qquad \textbf{.......................................(1.41)}$$

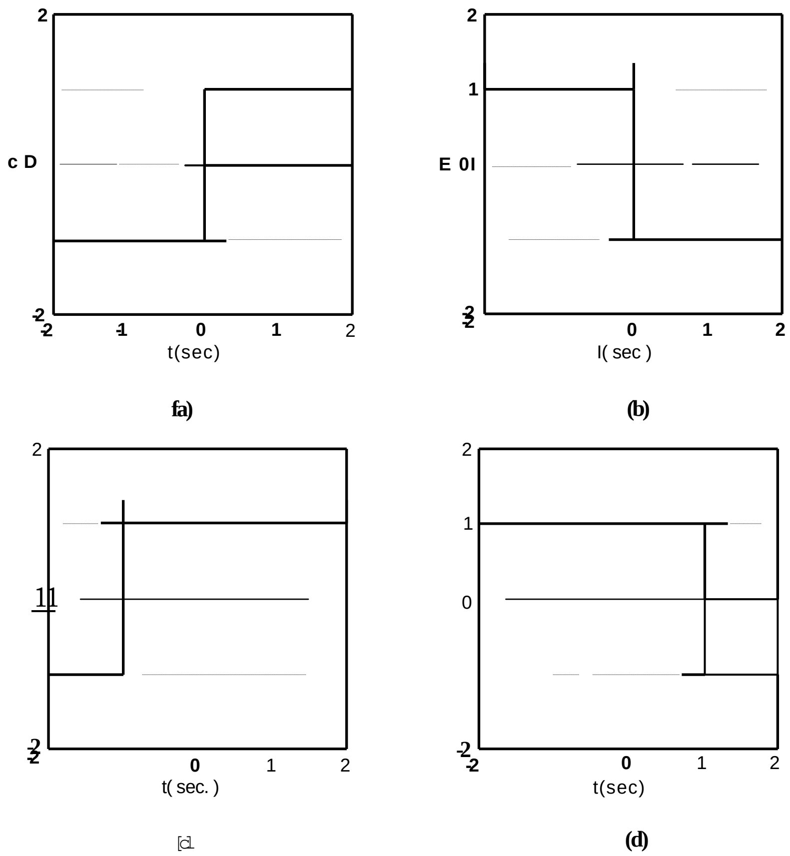


Fig 1.19 Plot of the unit signum function along with a few of its transformations

1.5 System viewed as interconnection of operation:

This article is dealt in detail again in chapter 2/3. This article basically deals with system connected in series or parallel. Further these systems are connected with adders/subtractor, multipliers etc.

1.6 Properties of system:

In this article discrete systems are taken into account. The same explanation stands for continuous time systems also.

The discrete time system:
The discrete time system is a device which accepts a discrete time signal as its input, transforms it to another desirable discrete time signal at its output as shown in figure **1.20**

Fig 1.20 DT system

Stability

A system is stable if 'bounded input results in a bounded output'. This condition, denoted by BIBO, can be represented by:

$$\sum_{n=-\infty} |x[n]| < \infty \qquad \sum_{n=-\infty} |y[n]| < \infty \quad \textit{for all} \quad n$$

(1.42)

Hence, a finite input should produce a finite output, if the system is stable. Some examples of stable and unstable systems are given in figure 1.21

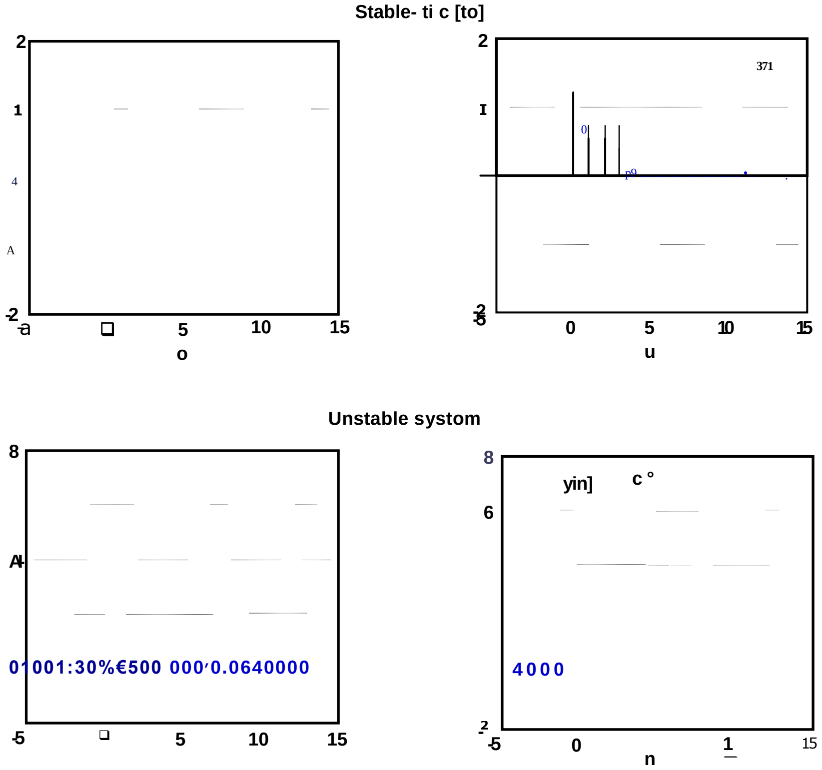


Fig 1.21 Examples for system stability

Memory

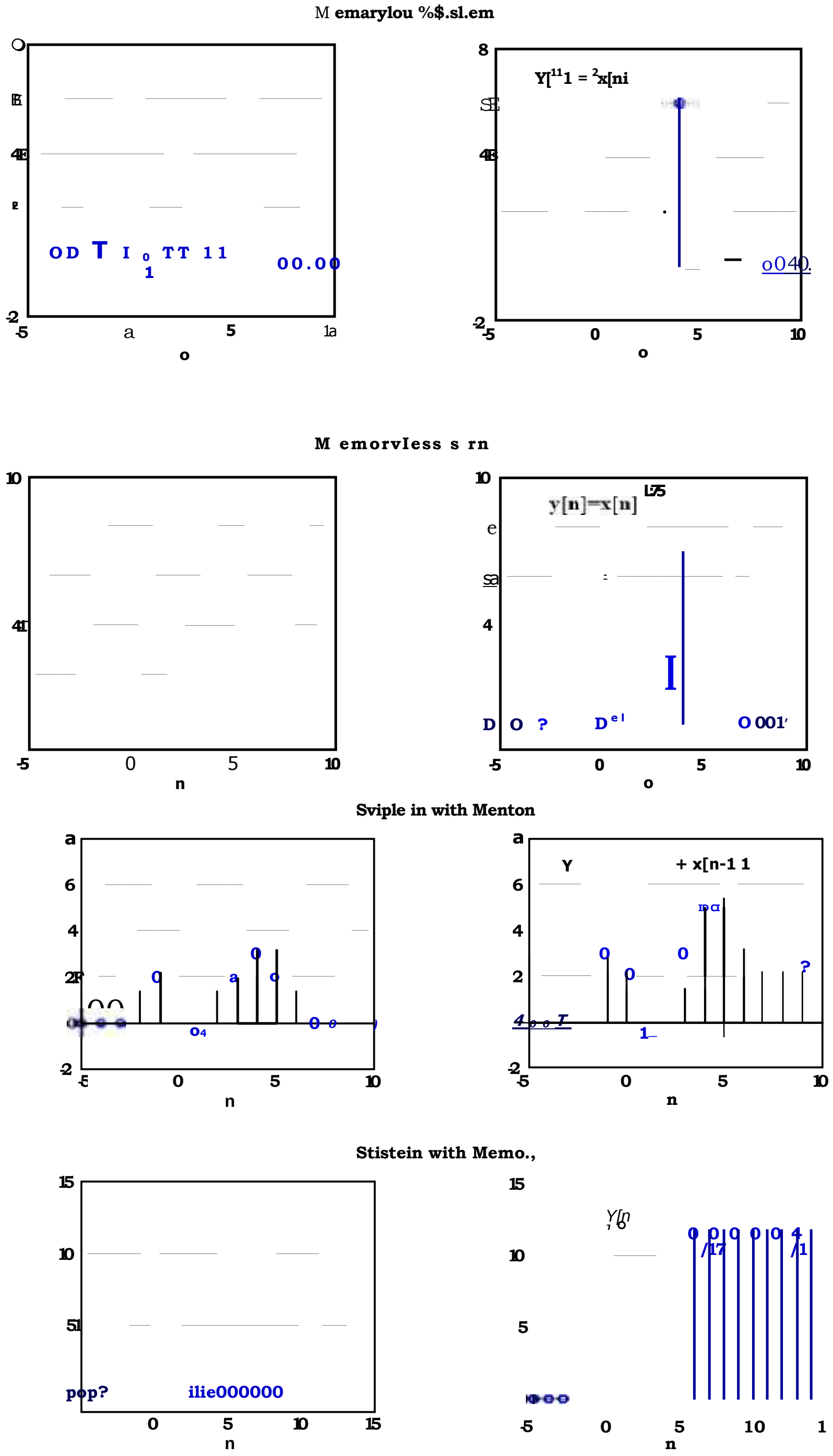
The system is memory-less if its instantaneous output depends only on the current input. In memory-less systems, the output does not depend on the previous or the future input.

Examples of memory less systems:

$y[n] = LaLtrj$

$y[n] = ax[n]$

$4/11 = a_1 + a_i v[n] + a_2 v^2 + a_3 v [ni$



Causality:

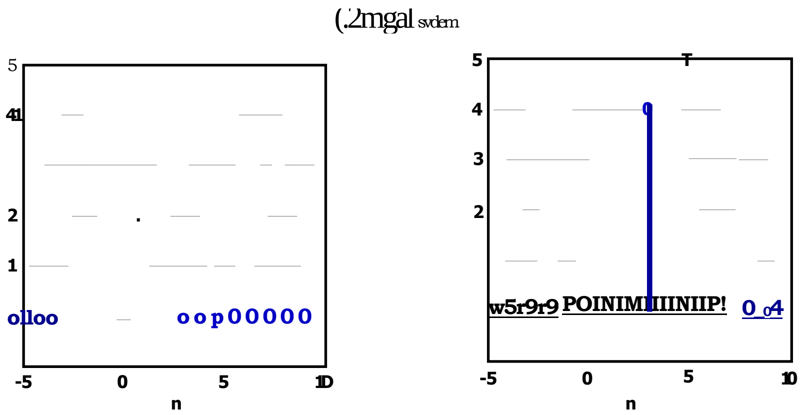
A system is causal, if its output at any instant depends on the current and past values of input. The output of a causal system does not depend on the future values of input. This can be represented as:

y[n] CIF *x[m]* *Efor m* Chz

For a causal system, the output should occur only after the input is applied, hence,

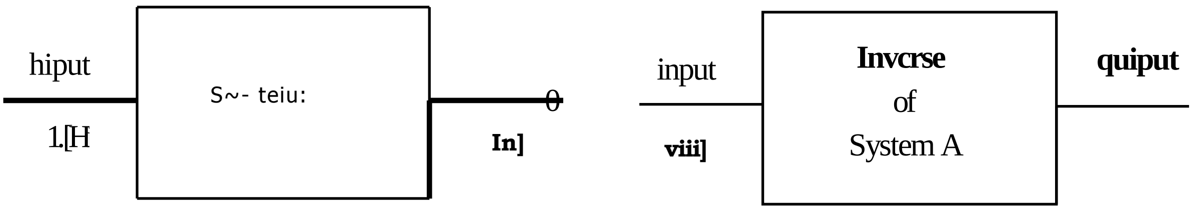
x[n] □ □ *Tor n* □ □ 0 implies *y[n]* □ □ *Tor n* □ □ 0

All physical systems are causal (examples in figure 7.5). Non-causal systems do not exist. This classification of a system may seem redundant. But, it is not so. This is because, sometimes, it may be necessary to design systems for given specifications. When a system design problem is attempted, it becomes necessary to test the causality of the system, which if not satisfied, cannot be realized by any means. **Hypothetical examples** of non-causal systems are given in figure below.



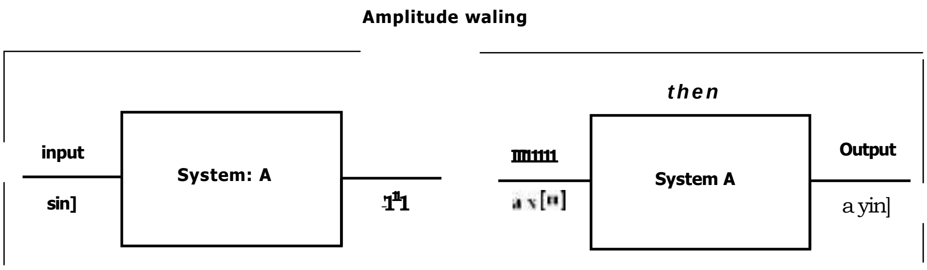
Invertibility:

A system is invertible if,

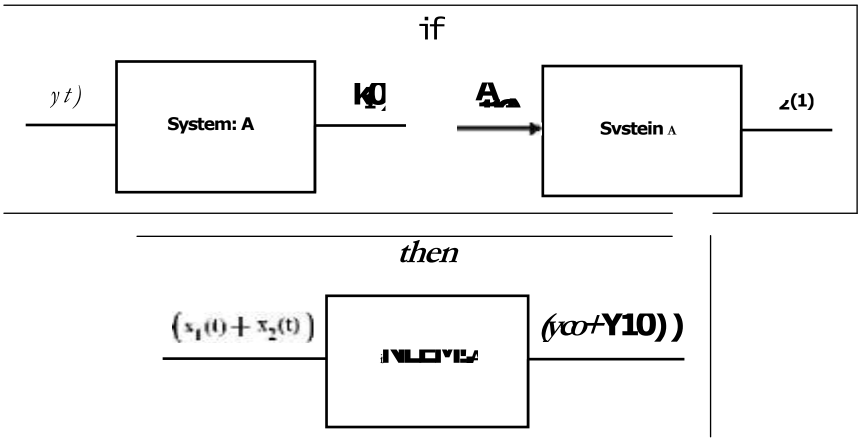


Linearity:

The system is a device which accepts a signal, transforms it to another desirable signal, and is available at its output. We give the signal to the system, because the output is s



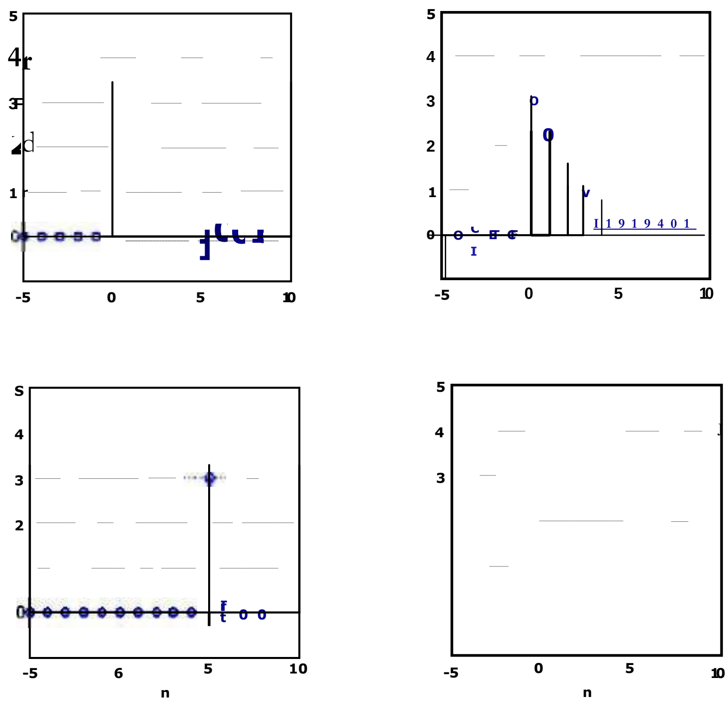
Superposition principle



Time

A system is time invariant, if its output depends on the input applied, and not on the time of application of the input. Hence, time invariant systems, give delayed outputs for delayed inputs.

Given input-uutput relation of Taue invariant nritrol



Recommended Questions

1. What are even and Odd signals
2. Find the even and odd components of the following signals
   a. x(t) = cos t + **sint +sintcost**
   b. ***x(t)*** **+1+ $3t^2$ + $5t^3$ + $9t^4$**
   c. x(t)+ (1+ t^3)cos$t^3$10t
3. What are periodic and A periodic signals. Explain for both continuous and discrete cases.
4. Determine whether the following signals are periodic. If they are periodic find the fundamental period.
   a. x (t) (c os(27ct) $)^2$
   b. x(n) cos(2n)
   c. x(n) cos 27in
5. **Define energy and power of a signal for both continuous and discrete case.**
6. **Which of the following are energy signals and power signals and find the power or energy of the signal identified.**

t, 0 1
a. x(t)=42—t, 1 2
Lin otherwise

n, 0<n<5
b. x(n)=X10—n, 5<_n<_10
Li-+ otherwise

15cosnt —0.5 0.5
c. x(t) H
LC

sinnn, —4<n<4
d. **x(n)**
LC otherwise

UNIT **2: Time-domain representations for LTI systems —1** **Teaching hours: 6**

Time-domain representations for LTI systems – 1: Convolution, impulse response representation, Convolution Sum and Convolution Integral.

TEXT BOOK

Simon Haykin and Barry Van Veen "Signals and Systems", John Wiley & Sons, 2001.Reprint 2002

REFERENCE BOOKS

1. **Alan V Oppenheim, Alan S, Willsky and A Hamid Nawab,** "Signals and Systems" Pearson Education Asia / PHI, 2nd edition, 1997. Indian Reprint 2002

2. **H. P Hsu, R. Ranjan,** "Signals and Systems", Scham's outlines, TMH, 2006

3. **B. P. Lathi,** "Linear Systems and Signals", Oxford University Press, 2005

4. **Ganesh Rao and Satish Tunga,** "Signals and Systems", Sanguine Technical Publishers, 2004

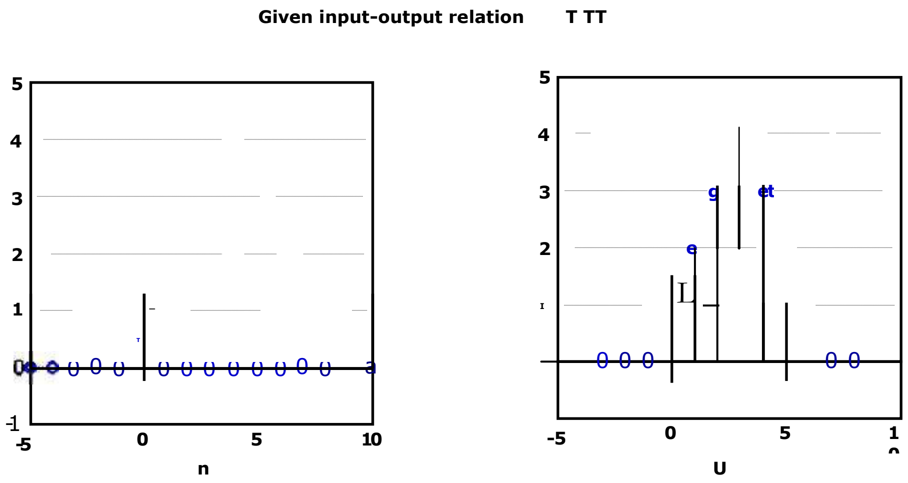
UNIT 2
Time-domain representations for LTI systems – 1

2.1 Introduction:

The Linear time invariant (LTI) system:

Systems which satisfy the condition of linearity as well as time invariance are known as linear time invariant systems. Throughout the rest of the course we shall be dealing with LTI systems. If the output of the system is known for a particular input, it is possible to obtain the output for a number of other inputs. We shall see through examples, the procedure to compute the output from a given input-output relation, for LTI systems.

Example – I:



2.1.1 Convolution:

A continuous time system as shown below, accepts a continuous time signal x(t) and gives out a transformed continuous time signal y(t).

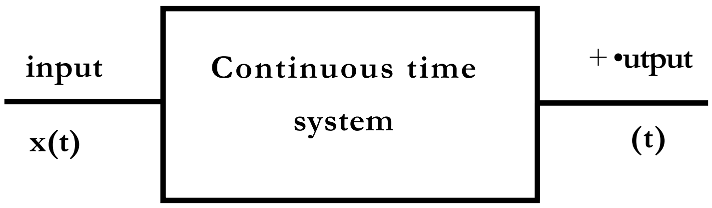


Figure 1: The continuous time system

Some of the different methods of representing the continuous time system are:

i) Differential equation
ii) Block diagram
iii) Impulse response
iv) Frequency response
v) Laplace-transform

vi) Pole-zero plot

It is possible to switch from one form of representation to another, and each of the representations is complete. Moreover, from each of the above representations, it is possible to obtain the system properties using parameters as: stability, causality, linearity, invertibility etc. We now attempt to develop the convolution integral.

2.2 Impulse Response

The impulse response of a continuous time system is defined as the output of the system when its input is an unit impulse, 8 *(t)* . Usually the impulse response is denoted by *h(t)* .

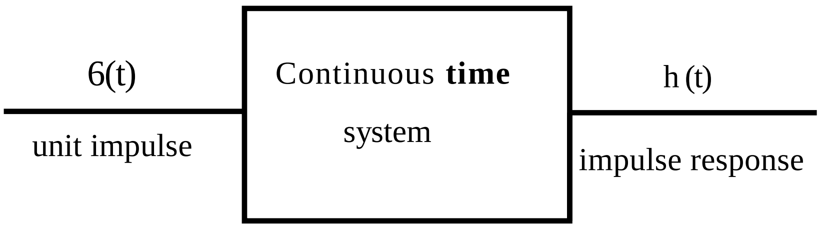
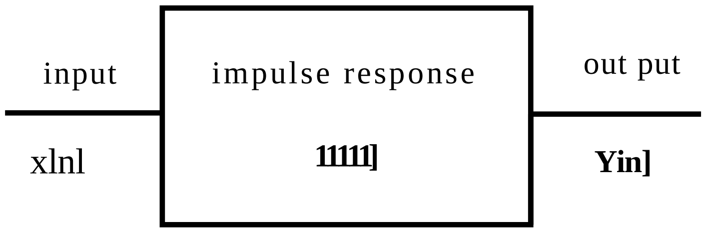
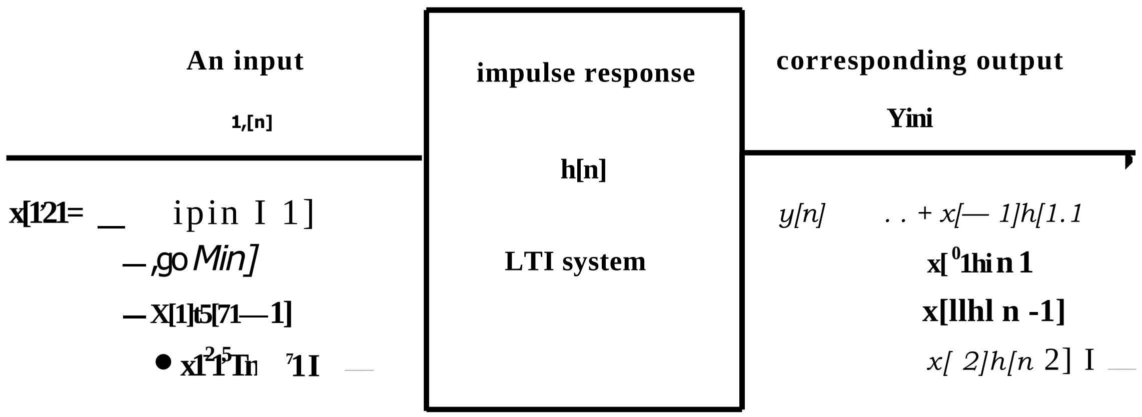


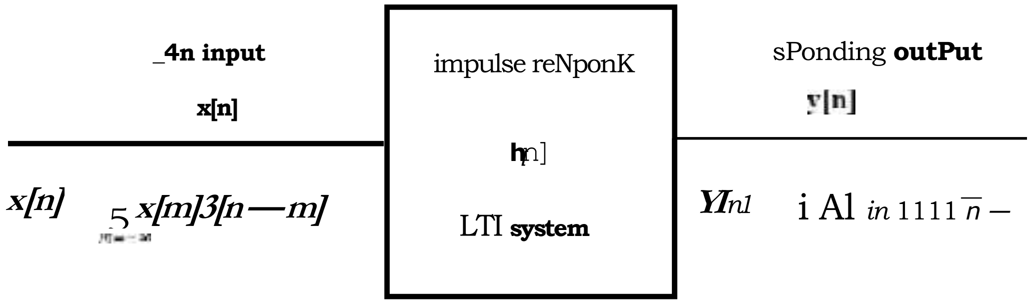
Figure 2: The impulse response of a continuous time system

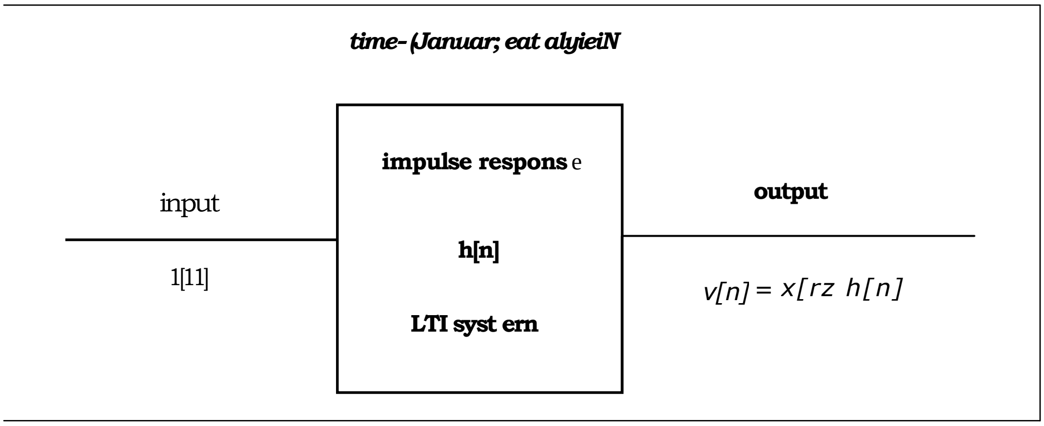
2.3 Convolution Sum:

We now attempt to obtain the output of a digital system for an arbitrary input x[n], from the knowledge of the system impulse response h[n].









\fit] 4 / 1] * h[ii]

Methods of evaluating the convolution sum:

Given the system impulse response h[n], and the input x[n], the system output y[n], is given by the convolution sum:

\ =

Problem:
To obtain the digital system output y[n], given the system impulse response h[n], and the system input x[n] as:

hii=11. -1.5. 3]

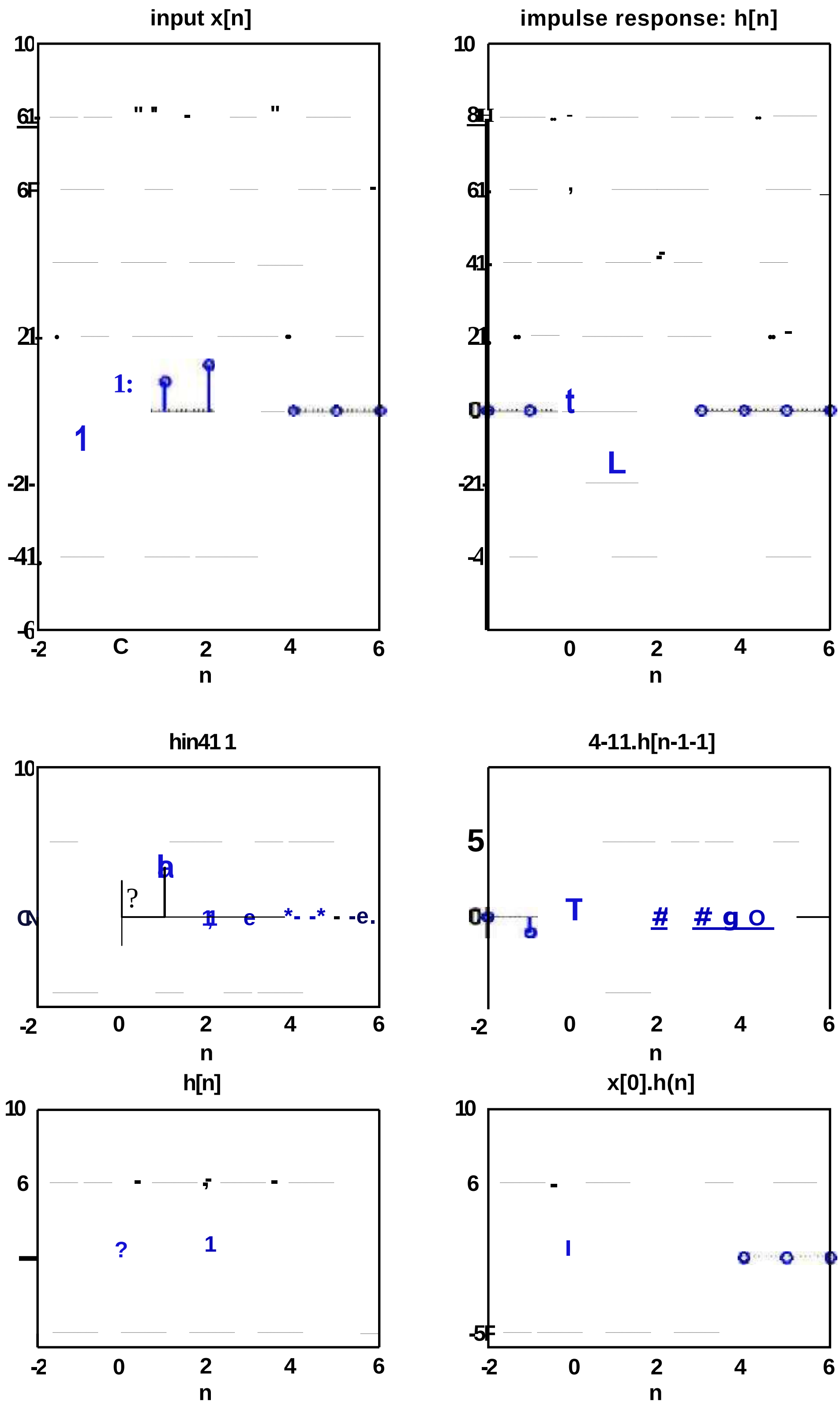
4.3.2] , [-1 , , I
1

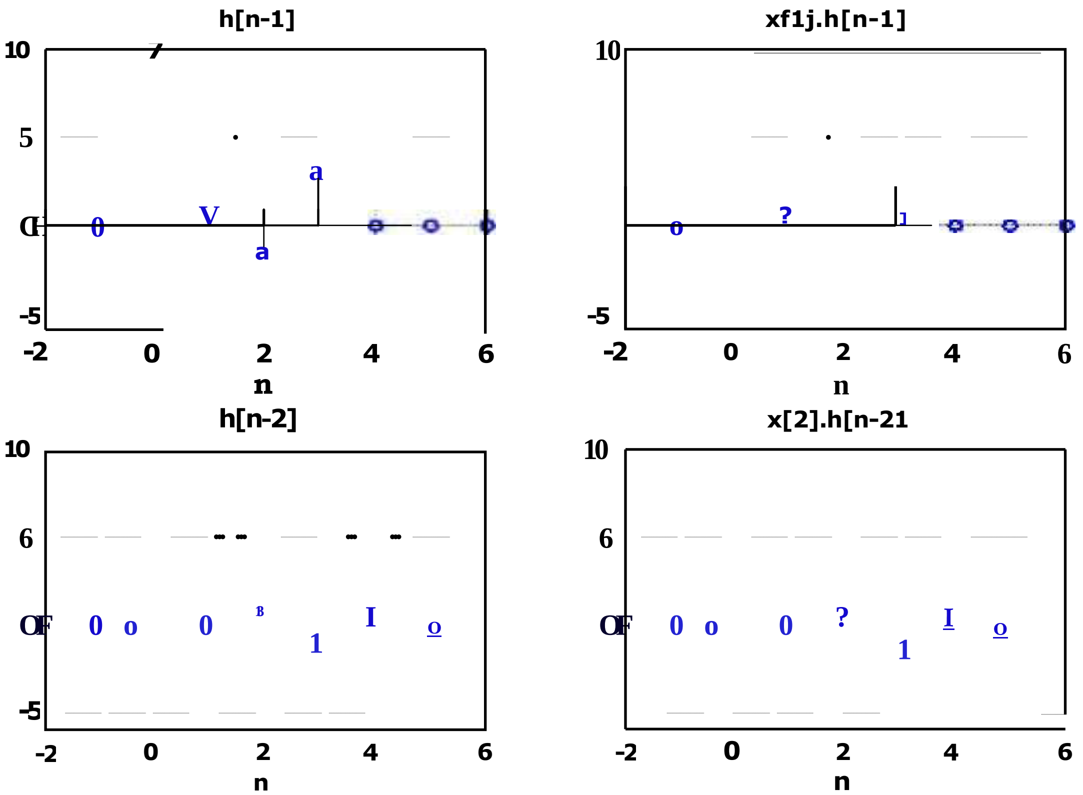
-1 4 -5.95 7.55 0.525 3.75

1. Evaluation as the weighted sum of individual responses

The convolution sum of equation (...), can be equivalently represented as:

y[n] 14 lJi[n 1] la[O]h[n] Dr[l]h[n 1]





Convolution as matrix multiplication:

Given

$x[ii] = [x_i \quad x_i$ starting from N.

and

$h[n] = [h_1 \quad h_m]$ starting from N_w

Step 1: Length of convolved sequence is NUM (L.+M-1)

Step 2: The convolved sequence starts at i $N_x + N_H$

Step 3: The convolution is given by the following matrix multiplication

$$\begin{bmatrix} Ail \\ y[i+1] \\ vri+21 \\ y[i+3] \\ y[i+4] \\ yU+5] \\ \\ \end{bmatrix} \begin{bmatrix} x_i & 0 & . & 0 \\ & x_i & . & 0 \\ & & . & 0 \\ & & & 0 \\ & . & . & \\ x_L & \bullet & . & \\ 0 & 0 & . & x_L \end{bmatrix} \begin{bmatrix} h_i \\ \\ h_m \end{bmatrix} \begin{bmatrix} h_1 & 0 & . & 0 \\ & h_i & . & 0 \\ & 2 & & 0 \\ . & & . & 0 \\ . & . & . & h_1 \\ & 214 & & \\ 0 & 0 & . & h_l \end{bmatrix} \begin{bmatrix} x_i \\ x_2 \\ \\ x_L \end{bmatrix}$$

The dimensions of the above matrices are:

$[NU \quad b \quad 1] = [NUM \quad by \quad M][M \quad by \quad 1] = [NUM \quad by \quad L][L \quad b \quad]$

For the given example:

x[n] is of length L=4, and starts at N_x= -1

h[n] is of length M=3 and starts at N_h 0

Step 1: Length of convolved sequence is NUM (L+M-1)=6

Step 2: The convolved sequence starts at i=(-1+0(-1)

$$\begin{bmatrix} y[-1] \\ y[0] \\ \\ y[2] \\ Y[3] \\ [4] \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 2.5 & -1 & 0 \\ 0.8 & 2.5 & -1 \\ 1.25 & 0.8 & 2.5 \\ 0 & 1.25 & 0.8 \\ 0 & 0 & 1.25 \end{bmatrix} - - \begin{bmatrix} -1 \\ 4 \\ -5.95 \\ 7.55 \\ 0.525 \\ 3.75 \end{bmatrix}$$

or

$$\begin{bmatrix} y[-1] \\ v[0] \\ Y[1] \\ ).[2] \\ Y[3] \\ y[41 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1.5 & 1 & 0 & 0 \\ 3 & -1.5 & 1 & 0 \\ 0 & 3 & -1.5 & 1 \\ 0 & & 3 & -1.5 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2.5 \\ 0.8 \\ 1.25 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ -5.95 \\ 7.55 \\ 0.525 \\ 3.75 \end{bmatrix}$$

Evaluation using graphical representation:

Another method of computing the convolution is through the direct computation of each value of the output y[n]. This method is based on evaluation of the convolution sum for a single value of *n*, and varying *n* over all possible values.

$$y[n] = \sum_{m=-\infty} ic[m]1$$

Step 1: Sketch x[m]

Step 2: Sketch h[-m]

Situp 3: Compute y[0] using:

$$y[0] = \sum x[m]h[-m]$$

which is the 'sum of the product of the two signals x[in] & h [-in] '

Stu]) 4: Sketch h[1-m], which is right shift of h[m] by 1. Compute

Step 5: y[1] using:

= 4170[1—

which is the 'sum of the product of the two signals x[in] & h[1-m]'

Step 6: Sketch h[2-m], which is right shift of h[m] by 2.

Step 7: Compute y[2] using:

$$y[2] = \sum x[m]h[2 - m]$$

which is the 'sum of the product of the two signals x[m] & h[2-m]'

Step 8: Proceed this way until all possible values of y[n]. for positive 'n' are computed

Step 9: Sketch h[4-m], which is left shift of **h[-m]** by 1.

Step **10:** Compute y[4] using:

$$y[-1] = \sum x[m]h[-1 - m]$$

which is the 'sum of the product of the two signals x[in] & h[-1-m]'

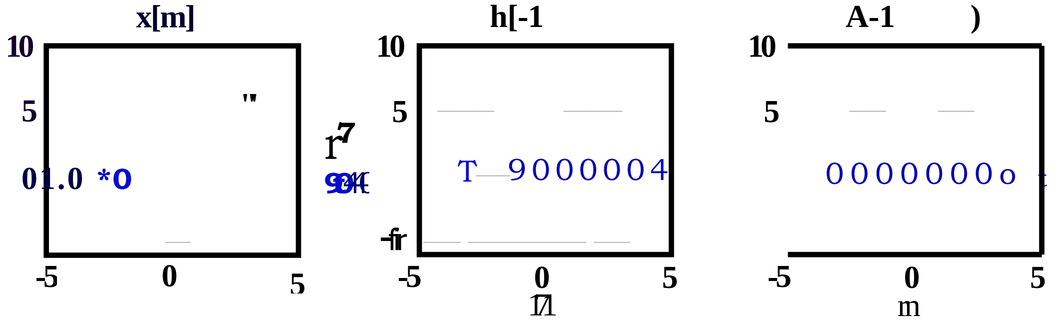
Step 11: Sketch h[-2-m], which is left shift of h[m] by 2. Compute

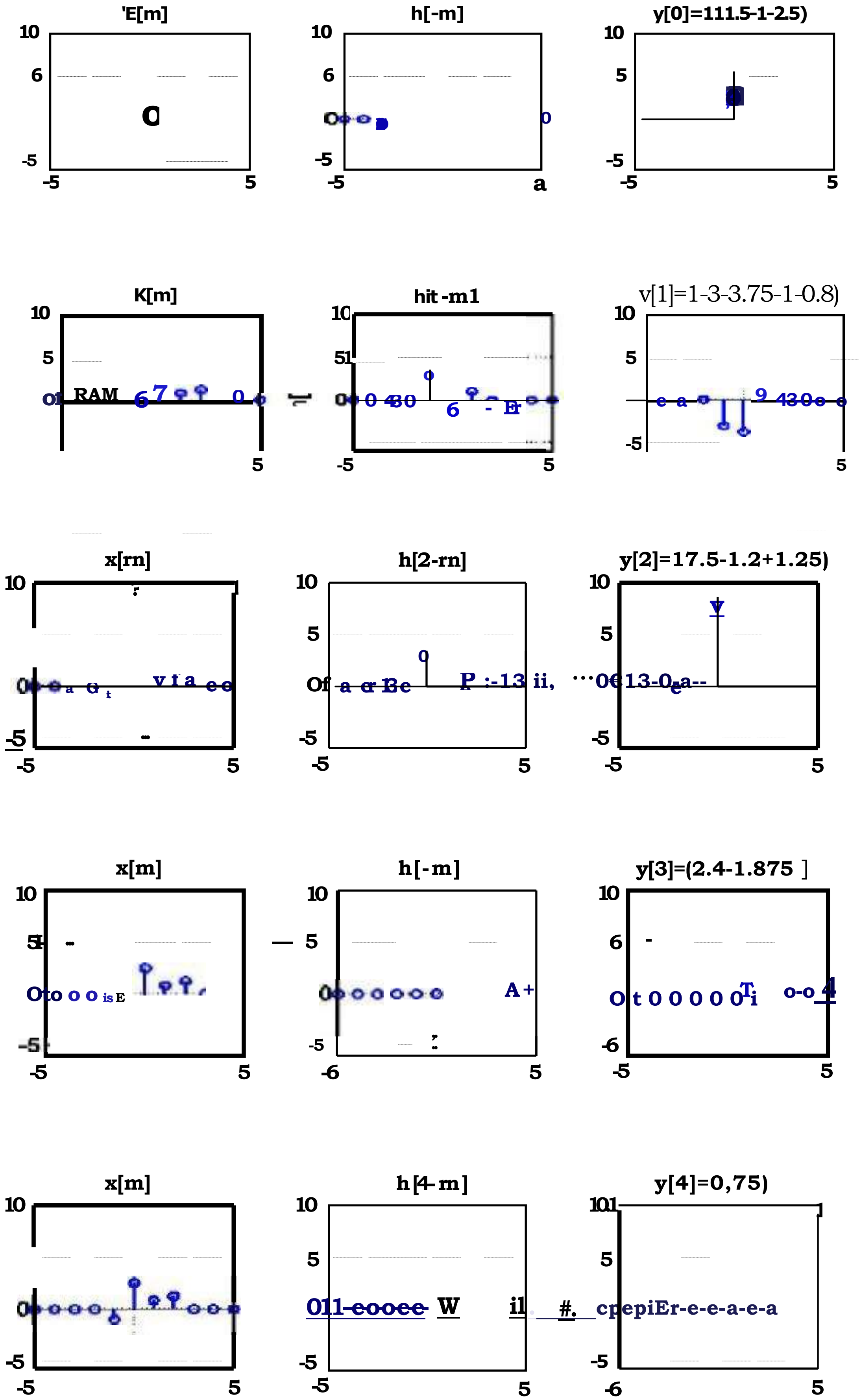
Step 12: y[-2] using:

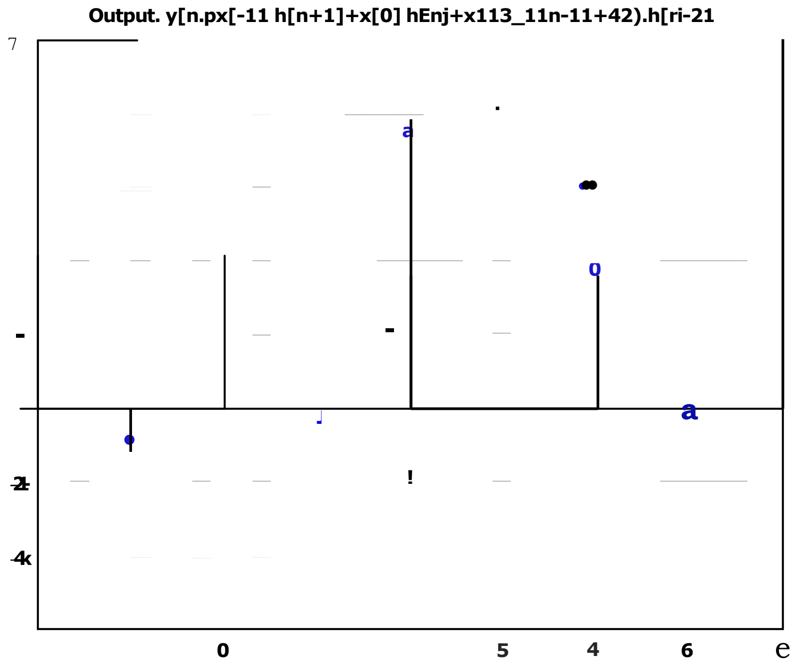
$$y[-2] = \sum_{m=-\infty} x[m]h[-2 - m]$$

which is the 'sum of the product of the two signals x[in] & h[-2-m]'

Step **13:** Proceed this way until all possible values of y[n], for negative 'if are computed




'E[m]
h[-m]
y[0]=111.5-1-2.5)
K[m]
hit -m1
v[1]=1-3-3.75-1-0.8)
x[rn]
h[2-rn]
y[2]=17.5-1.2+1.25)
x[m]
h[-m]
y[3]=(2.4-1.875]
x[m]
h[4-m]
y[4]=0,75)



Evaluation from direct convolution sum:

While small length, finite duration sequences can be convolved by any of the above three methods, when the sequences to be convolved are of infinite length, the convolution is easier performed by direct use of the 'convolution sum' of equation (...).

$$\text{since: } u[m] = \begin{cases} 0 & \text{for } m < 0 \\ 1 & \text{for } i ? 0 \end{cases}$$

$$u[n - m] = \begin{cases} & \text{for } (n - m) < 0 \\ 1 & \text{for } (n - m) ? 0 \end{cases}$$

$$\begin{cases} 0 & \text{for } (-m) < n \\ 1 & \text{for } (-in) ? n \end{cases}$$

$$\begin{cases} & \text{for } m > n \\ 1 & \text{for } m \end{cases}$$

Example: A system has impulse response $h[n] = \exp(-0.8)u[n]$. Obtain the unit step response.

Solution:

$$of\ n1 \quad \sum_{m=-\infty} h[m]x[m]$$

$$\sum_{m=-\infty} \{\exp(-0.8(m))u[m]\}\{u[n - m]\}$$

)T{exp $u[n - _m]$}

$m=0$

{ exp(-0.8(m))}

$m=0$

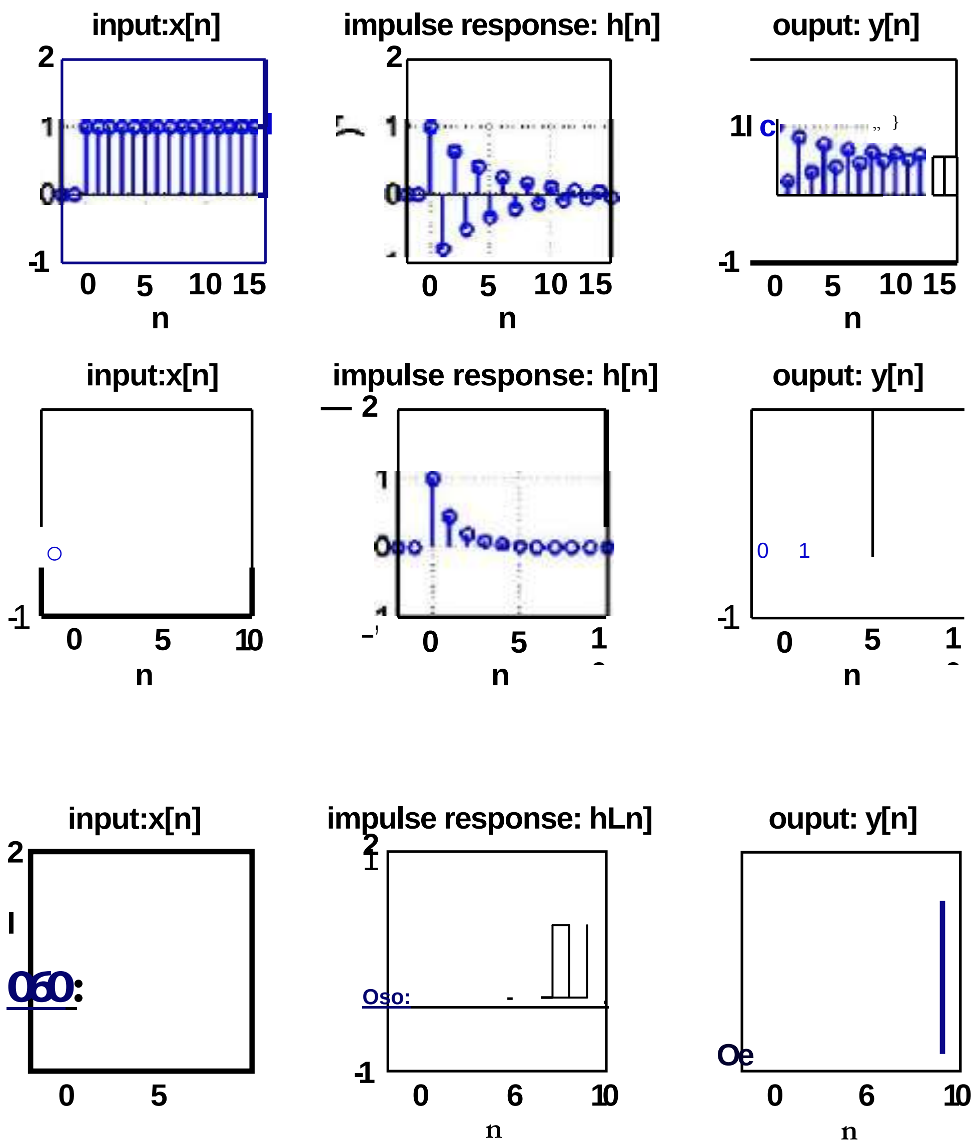
{ exp(-0.8(m)))

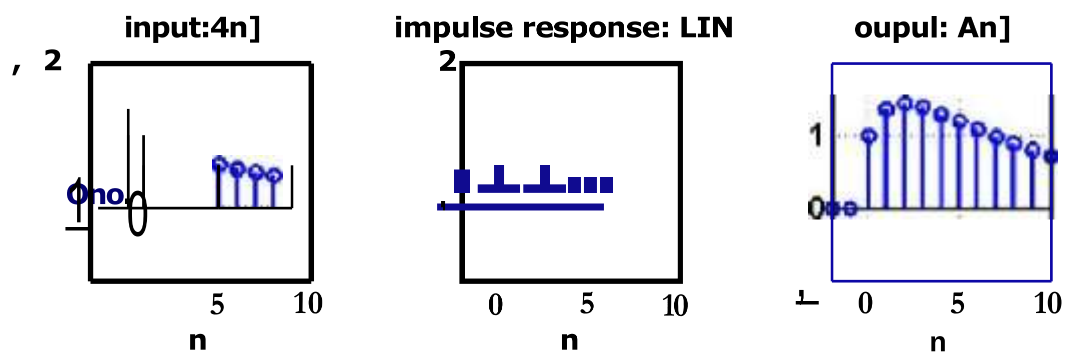
(1 — (-0.8))

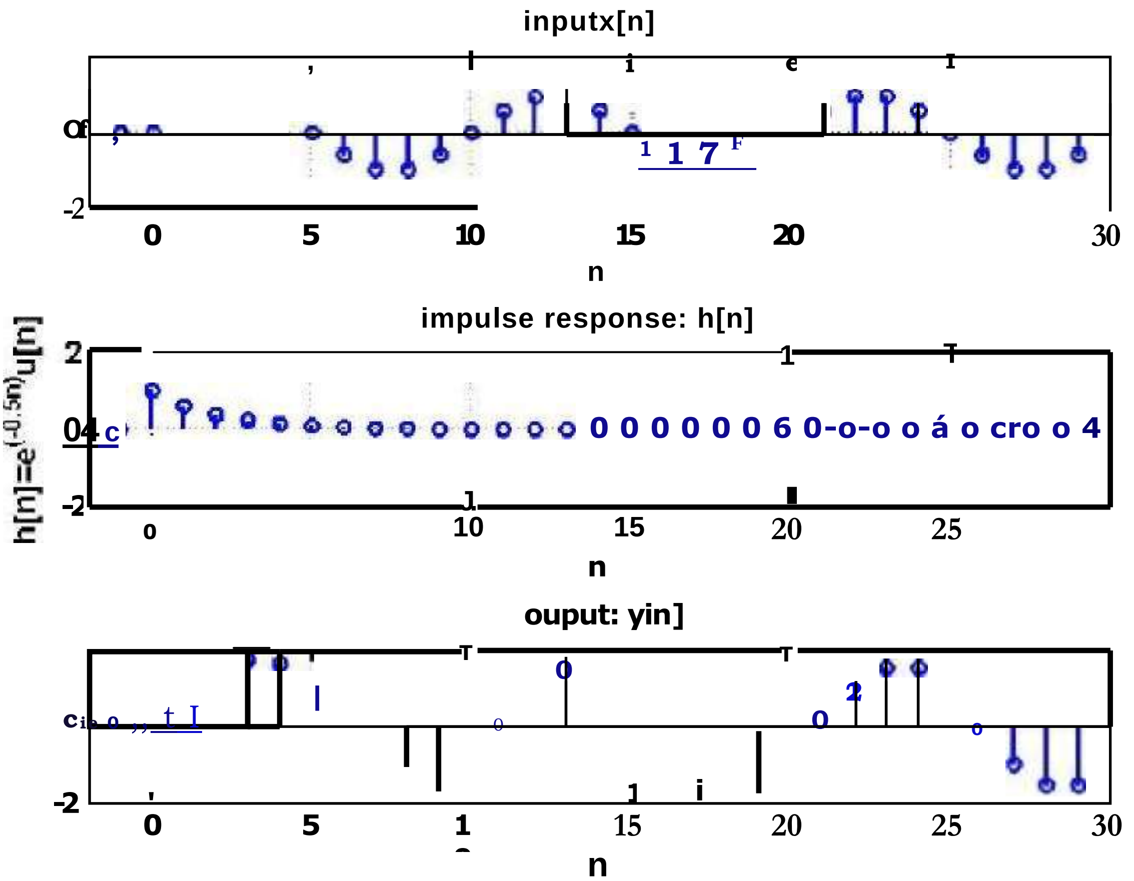
y [/ 1] = 1 (- .) u [n . – r r ~]

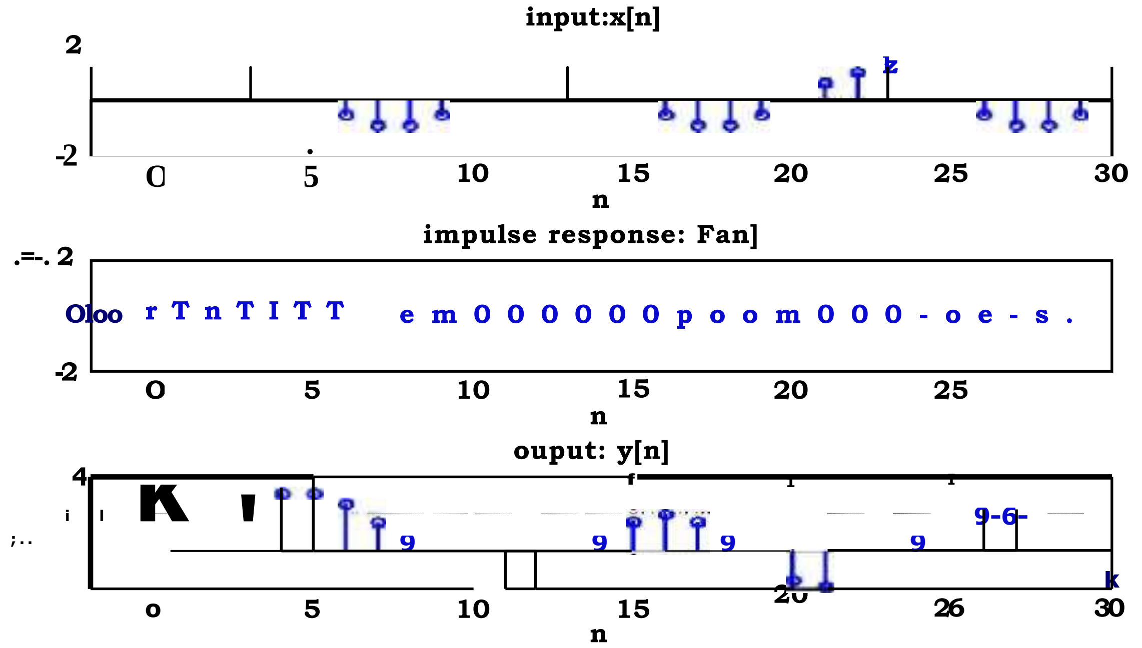
{exp (-0.8(n– in)} re[n –'n]1

m14



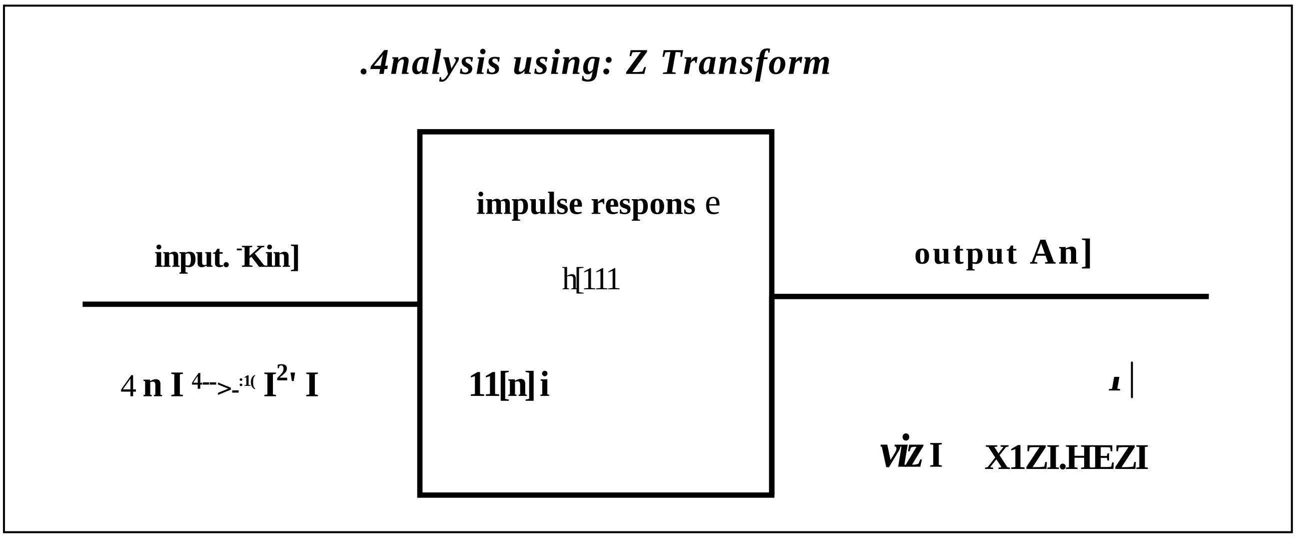

impulse response:
n
n
n


inputx[n]
n
impulse response: h[n]
$h[n]=e^{(-0.5n)}u[n]$
n
n


input:x[n]
n
n
ouput: y[n]
n

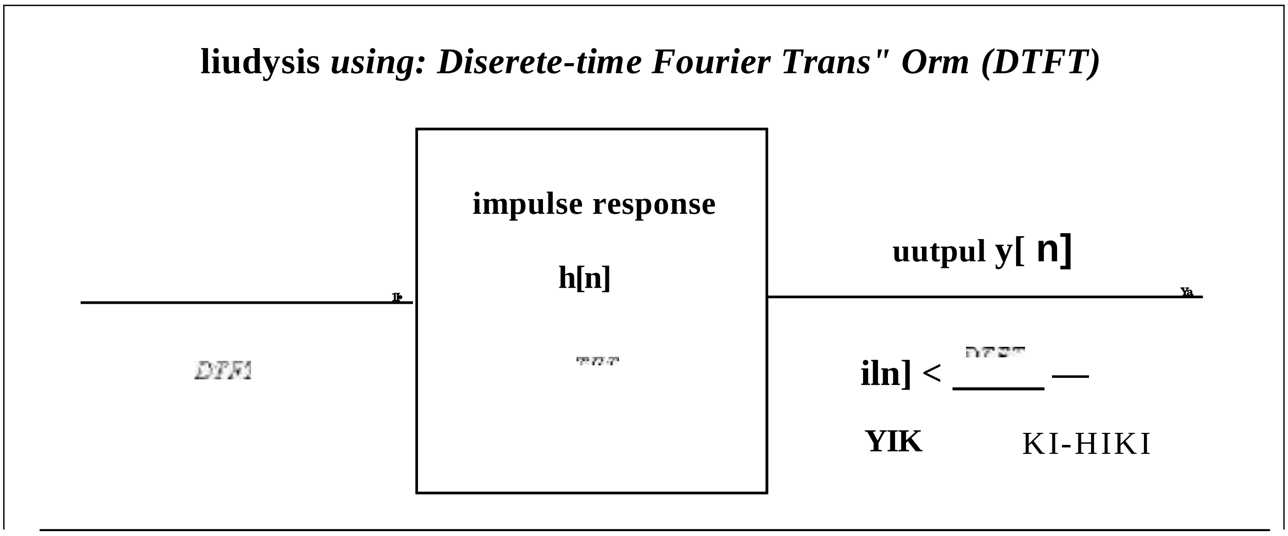
Evaluation from Z-transforms:

Another method of computing the convolution of two sequences is through use of Z-transforms. This method will be discussed later while doing Z-transforms. This approach converts convolution to multiplication in the transformed domain.



Evaluation from Discrete Time Fourier transform (DTFT):

It is possible to compute the convolution of two sequences by transforming them to the frequency domain through application of the Discrete Fourier Transform. This approach also converts the convolution operator to multiplication. Since efficient algorithms for DFT computation exist, this method is often used during software implementation of the convolution operator.



Evaluation from block diagram representation:

While small length, finite duration sequences can be convolved by any of the above three methods, when the sequences to be convolved are of infinite length, the convolution is easier performed by direct use of the 'convolution sum'

2.4 Convolution Integral:

We now attempt to obtain the output of a continuous time/Analog digital system for an arbitrary input x(t), from the knowledge of the system impulse response h(t), and the properties of the impulse response of an LTI system.

The output y(t) is given by, using the notation, y(tR{x(t)}.

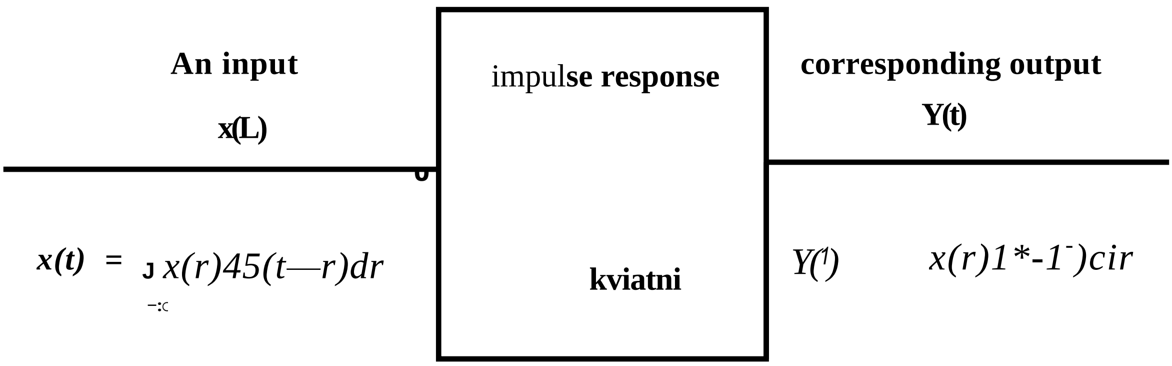
$$y(t) \quad Rfx(t)1$$

$$\{:f \ x(r)(50 \quad - \ r)d\text{-}r$$

$$f \quad (r) \qquad (t -r)\} \ dr$$

$$x(r)h(t -2)dr$$

$$x(t) \ h(t)$$



Methods of evaluating the convolution integral: (Same as Convolution sum)

Given the system impulse response h(t), and the input x(t), the system output y(t), is given by the convolution integral:

$$y(t) = \quad x(r)h(t = r)dAr$$

Some of the different methods of evaluating the convolution integral are: Graphical representation, Mathematical equation, Laplace-transforms, Fourier Transform, Differential equation, Block diagram representation, and finally by going to the digital domain.

Recommended Questions

1. Show that if x(n) is input of a linear time invariant system having impulse response h(n), then the output of the system due to x(n) is

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

2. Use the definition of convolution sum to prove the following properties
1. x(n) * [h(n)+g(n)]=x(n)*h(n)+x(n)*g(n) (Distributive Property)
2. x(n) * [h(n)*g(n)]=x(n)*h(n) *g(n) (Associative Property)
3. x(n) * h(n) =h(n) * x(n) (Commutative Property)
3. Prove that absolute summability of the impulse response is a necessary condition for stability of a discrete time system.
4. Compute the convolution y(t)= x(t)*h(t) of the following pair of signals:

(a) x(t) = -a < t < a ; 0 otherwise, h(t) = -a < t < a ; otherwise

(b) x(t) = 0 < t < T ; 0 otherwise, h(t) = 0 < t < 2T ; otherwise

(c) $x(t) = u(t - 1), h(t) = e^{-3t}u(t)$

5. Compute the convolution sum y[n] =x[n]* h[n] of the following pairs of sequences:

(a) x[n] - cg[n], h[ni - 2%4 -

(b) r[+]- u[n - N1, Mill- a' uf al, 0 <a

(c) x[n] ()" 14 n], Merl = St - $\frac{1}{2}\delta[n-1]$

6. Show that if y (t) =x(t)* h(t), then

yP(i) - x'(1)*c h()) -- xx((e01**

7. Let *y[n] = x[n]* h[n].* Then show that

I ft, j* Mil a y[n — rt,

8. Show that

Xani gIX2.[n] $\sum_{k=n_0}^{n_0+N-1}$ x ;rk]xjn - k]

for an arbitrary starting point *no.*

UNIT **3: Time-domain representations for LTI systems — 2** **Teaching hours: 7**

Time-domain representations for LTI systems — 2: properties of impulse response representation, Differential and difference equation Representations, Block diagram representations.

TEXT BOOK

Simon Haykin and Barry Van Veen "Signals and Systems", John Wiley & Sons, 2001.Reprint 2002

REFERENCE BOOKS

1. **Alan V Oppenheim, Alan S, Willsky and A Hamid Nawab,** "Signals and Systems" Pearson Education Asia / PHI, 2nd edition, 1997. Indian Reprint 2002

2. **H. P Hsu, R. Ranjan,** "Signals and Systems", Scham's outlines, TMH, 2006

3. **B. P. Lathi,** "Linear Systems and Signals", Oxford University Press, 2005

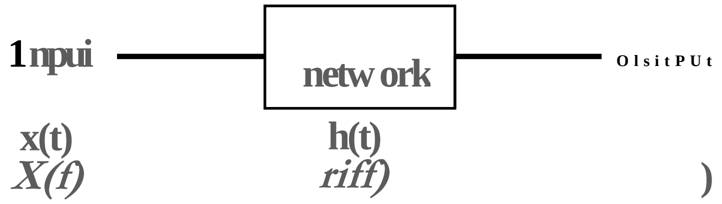
4. **Ganesh Rao and Satish Tunga,** "Signals and Systems", Sanguine Technical Publishers, 2004

UNIT 3:
Time-domain representations for LTI systems — 2

3.1 Properties of impulse response representation:

Impulse Response

Def. Linear system: system that satisfies superposition theorem.



For any system, we can define its impulse response as:

$$h(t) \quad y(t) \quad \text{whe} \quad (t) \quad 8(t)$$

For linear time invariant system, the output can be modeled as the convolution of the impulse response of the system with the input.

$$y(t) \quad \int_{-\infty}^{C} x(r)h(t \quad r)dr$$

For casual system, it can be modeled as convolution integral.

$$y(t) \quad \int_{0}^{\infty} x(r)h(t - r)dr$$

3.2 Differential equation representation:

General form of differential equation is

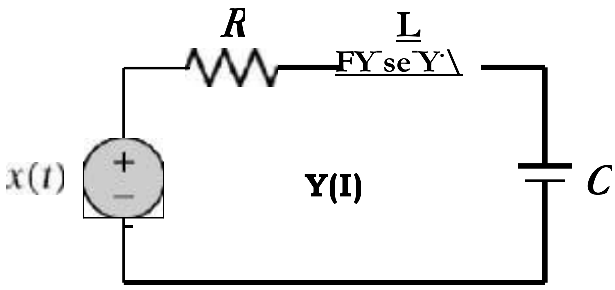
$$\sum^{V} \frac{d^k}{dt^k} Y(t) = \sum b_k \frac{d^k}{dt^k} x(t)$$

where *ak* and *bk* are coefficients, x(.) is input and *y(.)* is output and order of differential or difference equation is *(M,N).*

Example of Differential equation

• Consider the RLC circuit as shown in figure below. Let *x(t)* be the input voltage source and *y(t)* be the output current. Then summing up the voltage drops around the loop gives

$$Ry(t) + L\frac{d}{dt}y(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)\,d\tau = x(t)$$



3.3 Solving differential equation:

A wide variety of continuous time systems are described the linear differential equations:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

Just as before, in order to solve the equation for *y(t),* we need the ICs. In this case, the ICs are given by specifying the value of *y* and its derivatives 1 through N-1 at $t = 0^-$

- Note: the ICs are given at $t = 0^-$ to allow for impulses and other discontinuities at $t = 0$.

Systems described in this way are

linear time-invariant (LTI): easy to verify by inspection

Causal: the value of the output at time t depends only on the output and the input at times 0 < t < *t*

As in the case of discrete-time system, the solution *y(t)* can be decomposed into $y(t) = y_h(t) + y_p(t)$, where homogeneous solution or zero-input response (ZIR), $y_h(t)$ satisfies the equation

- The zero-state response (ZSR) or particular solution $y_p(t)$ satisfies the equation

$$y_p^{(N)}(t) + \sum_{i=0}^{N-1} a_i y_p^{(i)}(t) = \sum_{i=0}^{M} b_i x^{(i)}(t), \qquad t > 0$$

with ICs $y_p(0^-) = y_p^{(1)}(0^-) = \dots = y_p^{(N-1)}(0^-) = 0.$

Homogeneous solution (ZIR) for CT

- A standard method for obtaining the homogeneous solution or (ZIR) is by setting all terms involving the input to zero.

$$\sum_{i=0}^{N} a_i y^{(i)}(t) = 0, \qquad t > 0$$

and homogeneous solution is of the form

$$y_h(t) = \sum_{i=1}^{N} C_i e^{r_i t}$$

where r_i are the N roots of the system's characteristic equation

$$\sum_{k=0}^{N} a_k r^k = 0$$

and $C_1, \ldots, C_N$ are solved using ICs.

Homogeneous solution (ZIR) for DT

- The solution of the homogeneous equation

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0$$

is

$$y_h[n] = \sum_{i=1}^{N} c$$

where r_i are the N roots of the system's characteristic equation

$$\sum_{k=0}^{N} a_k r^{N-k} = 0$$

and , C_N are solved using ICs.

Example 1 (ZIR)

- Solution of

$$\frac{d^2}{dt^2} y(t) + 5 \frac{d}{dt} y(t) + 6y(t) = 2x(t) + \frac{d}{dt} x(t)$$

$$Yh(t) = c1\, e^{-3t} + c2e^{-2t}$$

- Solution of $y\,/7] - 9/16y[n - 2] = x[n - 1]$ is $yh[n] = c1\,(3/4)a + c2\,(-3/4)$

Example 2 (ZIR)

- Consider the first order recursive system described by the difference equation $y\,n] - py[n - 1] = x[n]$, find the homogeneous solution.
- The homogeneous equation (by setting input to zero) is $y\,n] - py\,n - 1] = 0$.
- The homogeneous solution for $N = 1$ is $yh[n] = cirr_i'$.
- $r1$ is obtained from the characteristics equation $r1 - p = 0$, hence $r1 = p$
- The homogeneous solution is $yh[n] = cip'$

Example 3 (ZIR)

- Consider the RC circuit described by $y(t) + RCly(t) = x(t)$
- The homogeneous equation is $y(t) + RC4\,.y(t) = \mathbf{0}$
- Then the homogeneous solution is

$$Yh(t) = ciei$$

where r1 is the root of characteristic equation $\mathbf{1} + RCr1 = u$

- This gives $r1 = -\frac{}{RC}$
- The homogeneous solution is

$$Yh(t) = ci^{\,0'}$$

Particular solution (ZSR)

- Particular solution or ZSR represents solution of the differential or dif ference equation for the given input
- To obtain the particular solution or ZSR, one would have to use the method of integrating factors.
- y_p is not unique.
- Usually it is obtained by assuming an output of the same general form as the input.
- If $x[n] = a^n$ then assume $y_p[n] =$ can and find the constant c so that $v_p[n]$ is the solution of given equation

1.1.3 Examples

Example 1 (ZSR)

- Consider the first order recursive system described by the differencE equation $y[n] - urn - 1] - x[n]$, find the particular solution when $x[n] -$ (1/2)n.
- Assume a particular solution of the form $y_p[n] = c_p(1/2)^{11}$
- Put the values of $y_p[n]$ and $x\ n]$ in the equation then we get c_pqr (D —

  pcpW n-1 =

- Multiply both the sides of the equation by (1 /2)n we get $c_p = 1/(1 -$ 2p).
- Then the particular solution is

$$Y_{pin} = \frac{1}{1 - 2p}(1)1$$

- For p = (1/2) particular solution has the same form as the homogeneous solution
- However no coefficient c_p satisfies this condition and we must assume a particular solution of the form $y_p[n] = c_p n(112)n$.
- Substituting this in the difference equation gives $c_{,,} n(l$ —2p) + 2Pcn = 1
- Using p = (1/2) we find that $c_p =$ 1

Example 2 (ZSR)

- Consider the RC circuit described by $y(t)$ -F RC 1)(0 — $x(t)$
- Assume a particular solution of the form $y_p(t)$ cl cos(tu0t)+c2 sin(wüt).
- Replacing $y(t)$ by $y_p(t)$ and $x(t)$ by cos(coot) gives

ct cos(coot) +c2 sin(coot) —Moog sin(coot) + *Mo0c2cos(coot)* — cos (coot)

- The coefficients c1 and c2 are obtained by separately equating the coefficients of cos(coot) and sin(coot), gives

$$\frac{1}{1+(ROD_0)2} \quad and \quad c2 \frac{RCCoo}{1+(Ra0_0)^2}$$

- Then the particular solution is

$$YD(t) \frac{1}{1+(R0.00)2} \cos(c)ot) + \frac{RCcoo}{1+(RC030)2} \sin(c)ot)$$

Complete solution

- Find the form of the homogeneous solution *yh* from the roots of the characteristic equation
- Find a particular solution y_p by assuming that it is of the same form as the input, yet is independent of all terms in the homogeneous solution
- Determine the coefficients in the homogeneous solution so that the complete solution y— $y_h \pm y_p$ satisfies the initial conditions

3.4 Difference equation representation:

- A wide variety of discrete-time systems are described by linear difference equations:

$$y[n] + X \sum_{k=1} a_k y[n - k] - \sum_{k=C}^{m} b_k x[n - k], \qquad n - 0,1,2,$$

where the coefficients a1, , aN and .60, , bm do not depend on n. In order to be able to compute the system output, we also need to specify the initial conditions (ICs) y[-1],y —2] y —N

- Systems of this kind are

— linear time-invariant (LTD: easy to verify by inspection

— causal: the output at time n depends only on past outputs An — 1],. , $y[n - N$ and on current and past inputs $x[n]$, $x[n$. ,$x[n$ M]

- Systems of this kind are also called Auto Regressive Moving-Average (ARMA) filters. The name comes from considering two special cases.
- auto regressive (AR) filter of order N, $AR(N)$: bo — — bm — 0

$$Y + \sum_{k=1}^{A} a_k Y in - \quad - 0 \qquad n - 0, 1, 2, ..$$

In the AR case, the system output at time n is a linear combination of /V past outputs; need to specify the ICs A-1], , $y[—N$.

- moving-average (MA) filter of order N, $AR(N)$: ao — — — 0

$$An] - \sum_{k=0}^{M} b \quad - k] \qquad n - 0, 1, 2, .$$

In the MA case, the system output at time n is a linear combination of the current input and M past inputs; no need to specify ICs.

- An ARMA(N. M) filter is a combination of both.
- Let us first rearrange the system equation

$$y[n] - - \sum_{k=1}^{N} a_k y[n \quad - k] + \sum_{k=o}^{M} b_k x[n \quad - k] \qquad n - 0,1, 2, \quad ...$$

- at $i1 = 0$

$$y[0] = -\sum_{k=1}^{N} \underbrace{a_k y[-k]}_{\text{depends on ICs}} + \underbrace{\sum_{k=0} b_k x[-k]}_{\text{depends on input } x[0] x[-M}$$

- at $n = 1$

$$y[1] = \sum_{k=1}^{N} a_k y[1-k] \pm \sum_{k=0} b_k x[1-k]$$

After rearranging

$$y[1] = -a_1 y[0] - \underbrace{\sum_{k=1}^{N-1} a_k y[1-k]}_{\text{depends on ICs}} + \underbrace{\sum_{k=0}^{M} b_k x[1-k]}_{\text{depends on input } x[1] \ldots x[1-M]}$$

. at $n = 2$

$$y[2] = \sum^{N} a_k y[2-k] \; \sum_{k=0}^{M} b_k x[2-k]$$

After rearranging

$$y[2] = -a_1 y[1] - a_2 y[0] - \underbrace{\sum^{N-1} a_k y[2-k]}_{\text{depends on ICs}} + \underbrace{\sum_{k=0}^{M} b_k x[2-k]}_{\text{depends on input } 44..42-M}$$

Example of Difference equation

- An example of **II** order difference equation is

$$y[n] + y[n-1] + \tfrac{1}{4} y[n-2] = x[n] \pm 2x[n-1]$$

- Memory in discrete system is analogous to energy storage in continuous system
- Number of initial conditions required to determine output is equal to maximum memory of the system

Initial Conditions

Initial Conditions summarise all the information about the systems past that is needed to determine the future outputs.

- In discrete case, for an N^{th} order system the N initial value are

$$y[-N], y[-N+1], \ldots, y[-1]$$

- The initial conditions for Nth-order differential equation are the values of the first N derivatives of the output

$$y(t)|_{t=0}, \frac{d}{dt}y(t)|_{t=0}, \frac{d^2}{dt^2}y(t)|_{t=0}, \ldots, \frac{d^{N-1}}{dt^{N-1}}y(t)|_{t=0}$$

Solving difference equation

- Consider an example of difference equation $y[n] + ay[n-1] = x[n]$, with $y[-1] = 0$ Then

$$\begin{aligned}
y[0] &= -ay[-1] + x[0] \\
y[1] &= -ay[0] + x[1] \\
&= -a(-ay[-1] + x[0]) + x[1] \\
&= a^2 y[-1] - ax[0] + x[1] \\
y[2] &= -ay[1] + x[2] \\
&= -a(a^2 y[-1] - ax[0] + x[1]) + x[2] \\
&= -a^3 y[-1] + a^2 x[0] - ax[1] + x[2]
\end{aligned}$$

and so on

- We get $y[n]$ as a sum of two terms:
$y[n] = (-a)^{n+1} y[-1] + \sum_{k=0}^{n} (-a)^k x[n-k], \quad n = 0, 1, 2, \ldots$

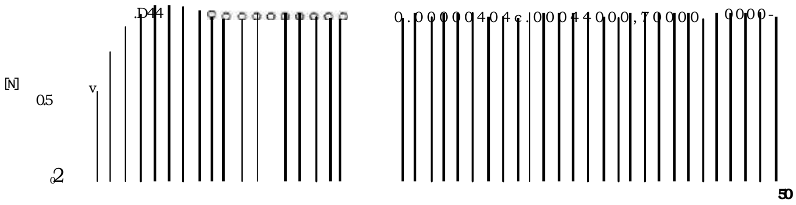
- First term $(-a)^{n+1} y[-1]$ depends on IC's but not on input

- Second term $_o$(—a)nix[i] depends only on the input, but not on the IC's
- This is true for any ARMA (auto regressive moving average) system: the system output at time n is a sum of the AR-only and the MA-only outputs at time 17
- Consider an ARMA (NT,M) systemyjn] — — I f^{a}1A ^{n} — 11+14M0 bPc[n — n], $n = 0, 1, 2, \ldots$ with the initial conditions y_-1], ___ ,y[—N].
- Output at time 12 is:

$$Y[n] = Y_h[n] \pm Y_p[n]$$

where $y_h[n]$ and $y_p[n]$ are homogeneous and particular solutions

- First term depends on IC's but not on input
- Second term depends only on the input, but not on the IC's
- Note that $y_h[n]$ is the output of the system determined by the ICs only (setting the input to zero), while $y_p[n]$ is the output of the system determined by the input only (setting the ICs to zero).
- $y_h[ri]$ is often called the zero-input response (ZIR) usually referred as homogeneous solution of the filter (referring to the fact that it is determined by the ICs only)
- $y_p[n]$ is called the zero-state response (ZSR) usually referred as particular solution of the filter (referring to the fact that it is determined by the input only, with the ICs set to zero).



Step response of a system

Figure 1.2: Step response

- Consider the output decomposition *yjn]* — + $y_p[n]$ of an ARMA *(N, M)* filter

$$An] = -\sum_{i=1}^{N} a_i y[n-i] + \sum_{i=0}^{M} b_i x[n-i], \qquad n = 0, 1, 2 \ldots$$

with the ICs y —1], ... , $y[-N]$.

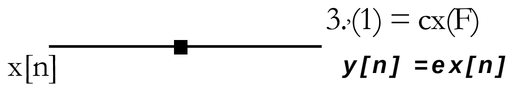
- The output of an ARMA filter at time n is the sum of the ZIR and the ZSR at time n.

Example of difference equation

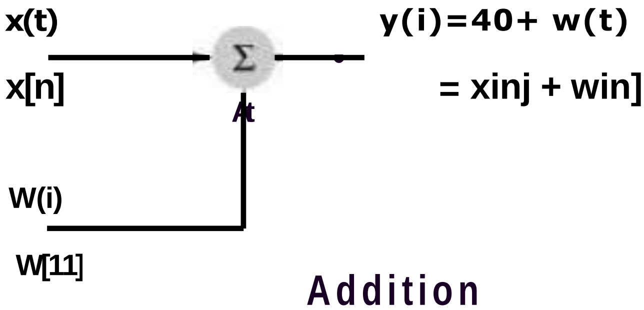
- example: A system is described by $y[n] - 1.143y[n-1] + \mathbf{0.4128y[n-2]} = 0.0675x[n] \pm 0.1349x[n-1] \pm 0.675x[n-2]$
- Rewrite the equation as $y[n] = \mathbf{1.143y[n-1]} - \mathbf{0.4128y}[n-2] \pm 0.0675x[n] \pm 0.1349x[n-1] \pm 0.675x[n-2]$

3.5 Block Diagram representation:

- A block diagram is an interconnection of elementary operations that act on the input signal
- This method is more detailed representation of the system than impulse response or differential/difference equation representations
- The impulse response and differential/difference equation descriptions represent only the input-output behavior of a system, block diagram representation describes how the operations are ordered
- Each block diagram representation describes a different set of internal computations used to determine the system output
- Block diagram consists of three elementary operations on the signals:
  - Scalar multiplication: $y(t) \;— cx(t)$ or $y[n] \;— x[n]$, where c is a scalar
  - Addition: $y(t) \;— x(t) + w(t)$ or $y[n] \;— x[n] + w[n]$.
- Block diagram consists of three elementary operations on the signals:
  - Integration for continuous time LTI system: $y(t) = \text{ft } x(ti)\, dit$
    Time shift for discrete time LTI system: $y[n] = x[n - 1]$
- Scalar multiplication: $y(t) \;— cx(t)$ or $y[n] \;— x[n]$, where c is a scalar

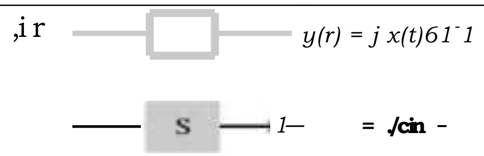


Scalar Multiplication



Addition

- Addition: $y(t) = x(t) + w(t)$ or $y[n] = x[n] + \_n]$
- Integration for continuous time LTI system: $y(t) = .1f \;. x(i)\, di$
  Time shift for discrete time LTI system: $y[n] = x[n - 1]$



Integration **and timeshifting**

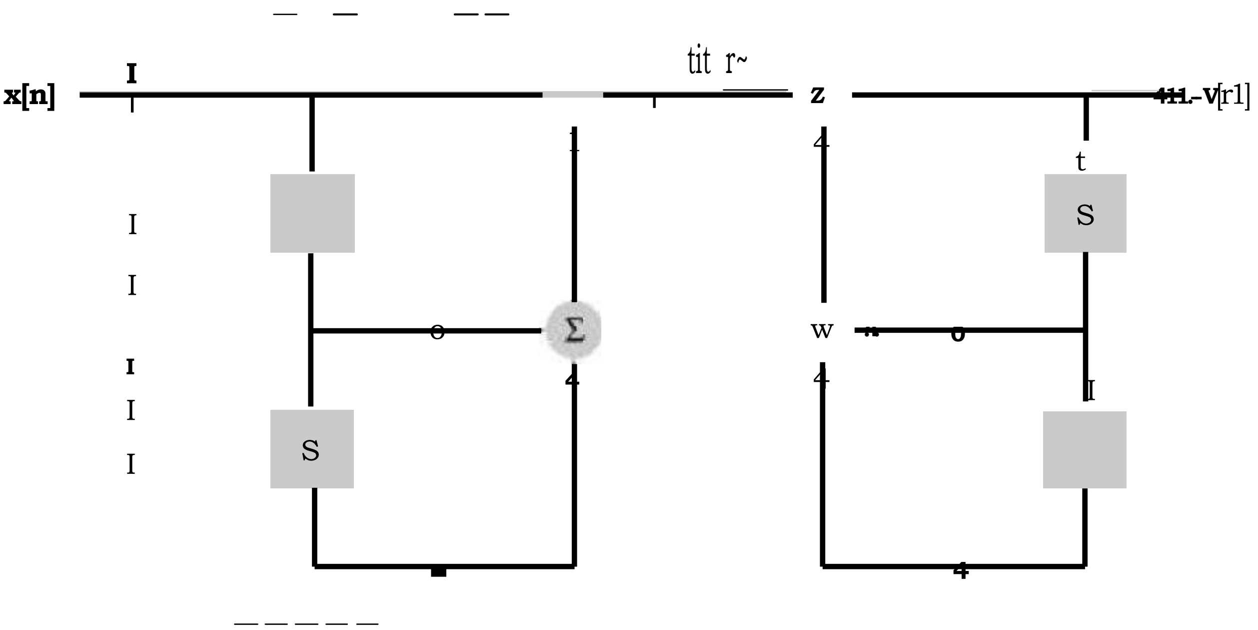


Figure 1.10: Example 1: Direct form I

Example 1

- Consider the system described by the block diagram as in Figure 1.10
- Consider the part within the dashed box
- The input $x[n]$ is time shifted by 1 to get $x[n-1]$ and again time shifted by one to get $x[n-2]$. The scalar multiplications are carried out and

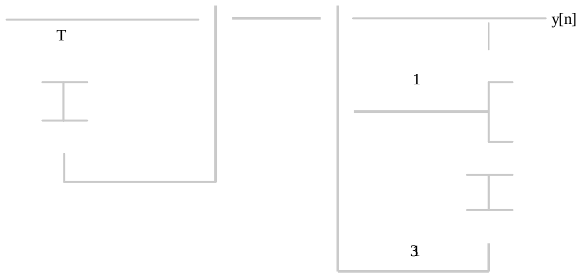


Figure 1.11: Example 2: Direct form I

they are added to get w[n] and is given by

$$w[n] — box[n] \pm b\ 1\ x[n — 1] \pm b2x[n — 2]$$

- Write $y[n]$ in terms of w[n] as input $y[n] — w[n] — al\ An— 1] — a2y[n — 2]$
- Put the value of $w[n]$ and we get $y[n] — —a1y[n — 1] — a234r— 2] \pm b0x[n]$
  $\pm bix[n — 1] \pm b2x[n — 2]$

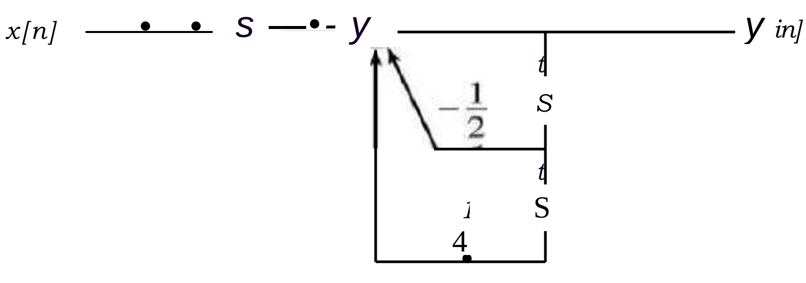
  and $y[n] + a\ iy[n — 1] \pm a2y[n — 2] — box[n] \pm bix[n — 1] \pm b2x[n— 2]$
- The block diagram represents an LTI system

Example 2

- Consider the system described by the block diagram and its difference equation is $y[n] \pm (1 / 2)\ y[n— 1] — (1 / 3)\ An— 3] — x[n] \pm 2\ x[n — 2]$

Example 3

- Consider the system described by the block diagram and its difference equation is $y[n] \pm (1 / 2)\ y[n — 1] + (1 / 4)y[n — 2] — x[n — 1]$



(13)

Figure 1.12: Example 3: Direct form **I**

- Block diagram representation is not unique, direct form II structure of Example **1**
- We can change the order without changing the input output behavior

Let the output of a new system be *f[n]* and given input *x[n]* are related by

$$f[n] \; — \; —al\, f[n \; — \; 1] \; — \; a2\, f[n — 2] \pm x[n]$$

- The signal *f[n]* acts as an input to the second system and output of second system is

$$n] — bo\, f[n] \pm fit\, f[n \; — \; 1] \pm b2\, f[n — 2]$$

- The block diagram representation of an LTI system is not unique

Continuous time

- Rewrite the differential equation

$$\sum_{k=0} ak \_\_ \text{-}_{,} k \; — \; \sum_{k=o}^{1V.} 1bk \frac{dk}{d\, tk}$$

as an integral equation. Let $1(°)(t) \; — \; v(t)$ be an arbitrary signal, and set

$$v^{(n)}(t) \; — \; \int_{/1}^{=} v^{(11\text{-}1)}(t)cl\text{-}c, \qquad n — 1.2\_3$$

where $v(n)(t)$ is the n-fold integral of $v(t)$ with respect to time

- Rewrite in terms of an initial condition on the integrator as

$$v^{00}(t) \; — \; \int_{o} v@i^{-1}) \; (T)cf\text{-}r \pm v(n^{)} \; (0) \; , \; n — 1, 2, 3,$$

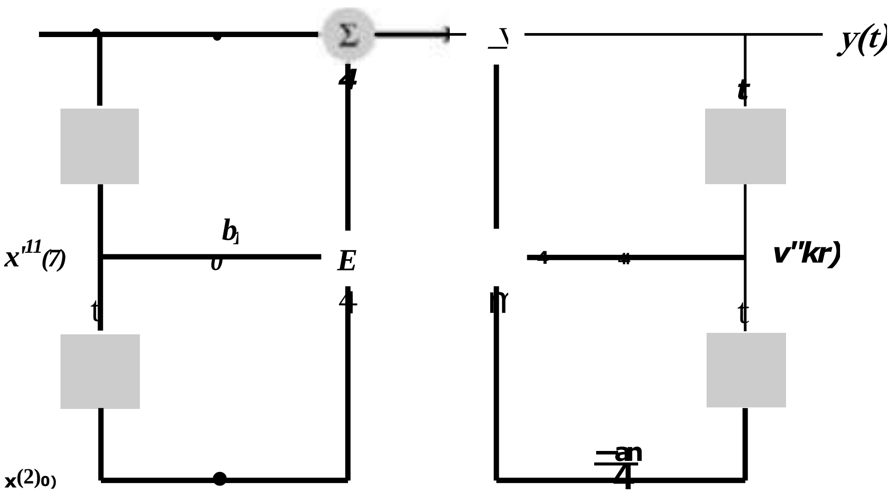
- If we assume zero ICs, then differentiation and integration are inverse operations, ie.

$$\frac{d}{dt} v'' \; — \; v^{(n-1)}(t), \; t > 0 \text{ and } n \; — \; 1, 2, 3, .\_$$

- Thus, if $N > Al$ and integrate N times, we get the integral description of the system

$$\sum k \_\_ {}_{n}N\; aky(N\text{–}k)(t) \; — \; I \; {}^{k} \_\_ 0Mbkx(N\text{-}k)\; (t)$$

- For second order system with a_o — 1, the differential equation can be

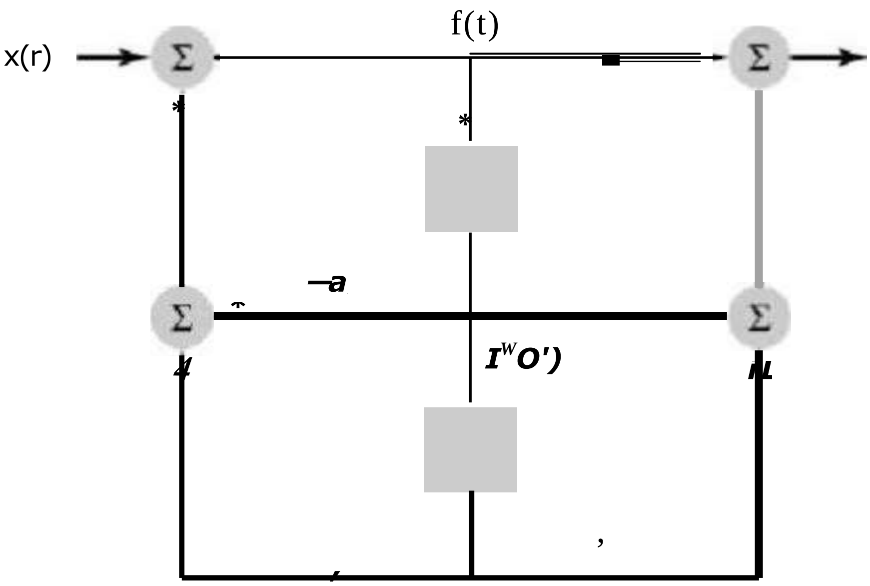


Direct form I structure

Figure 1.25: Direct form 1

written as

$$y(t) = -a_1 y^{(1)}(t) - a_0 y^{(2)}(t) + b_2 x(t) + a_1 x^{(1)}(t) + b_0 x^{(2)}(t)$$



Direct form 11 structu 1 e

Recommended Questions

1. Show that

(a) x(t) - x(r)*

(b) x(z) SO t_o) - t_o)*

x(t) u(t).. x(r)dr*

(d) x(t) u(t t,,)* $\int^{t-t_0}$ *x(r)dr*

By definition *(2.6)* and Eq. *(1.22) we* have

x(1).6(i) - 1 x(7)6(1* $\tau)\,d\tau = x(\tau)|_t$

By Eqs. ***(2.7)* and *(L22) we* have**

*x(1)*45(i - 1_0) 8(t* $t_0) * x(t) =$ *r t_o)x(t - 7) dr*

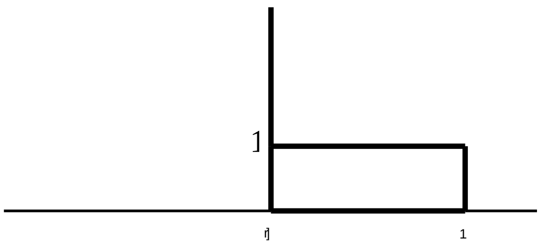
- x(1 - x(r -1_0)

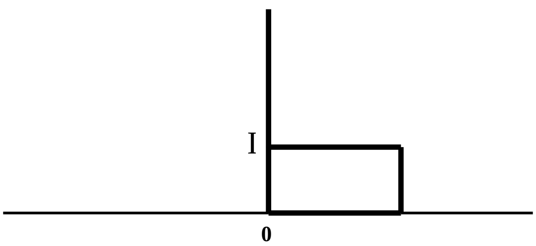
By Eqs. *(2.6)* and (if. /9) we have

*x(r)*u(r) - i x(7)u(i iS x(r) dr*

since u(r —r) . 7< t

2. Evaluate y (t) = x (t) * h(t), where x (t) and h (t) are shown in Fig. 2-6 (a) by analytical technique, and (b) by a graphical method.





2-6

3. Consider a continuous-time LTI system described by

4. $$\_V(O = 11X(1)) = \frac{1}{T}\int_{T/2} X(T)Ctr$$

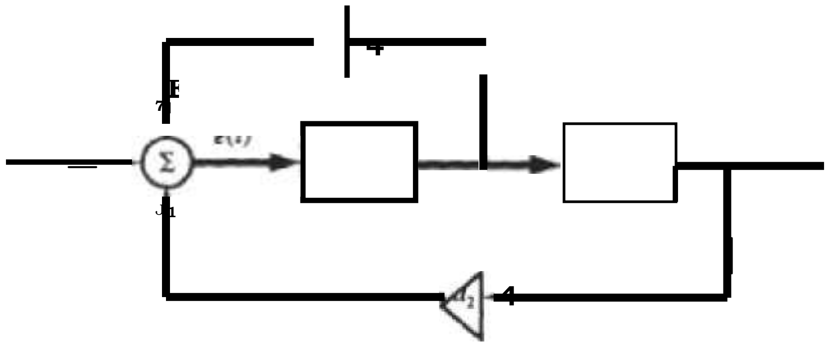
a. Find and sketch the impulse response h(t) of the system.
b. Is this system causal?

5. Let y (t) be the output of a continuous-time LTI system with input x(t) . Find the output of the system if the input is $x^1(t)$, where x^1 (t) is the first derivative of x(t)

6. Verify the BIBO stability condition for continuous-time LTI systems.

7. Consider a stable continuous-time LTI system with impulse response *h (t)* that is real and even. Show that Cos wt and sin *wt* are Eigen functions of this system with the same real Eigen value.

8. The continuous-time system shown in Fig. 2-19 consists of two integrators and two scalar multipliers. Write a differential equation that relates the output y(t) and the input x(t).



UNIT 4: Fourier representation for signals —1 **Teaching hours: 6**

Fourier representation for signals — 1: Introduction, Discrete time and continuous time Fourier series (derivation of series excluded) and their properties

TEXT BOOK

Simon Haykin and Barry Van Veen "Signals and Systems", John Wiley & Sons, 2001.Reprint 2002

REFERENCE BOOKS

1. **Alan V Oppenheim, Alan S, Willsky and A Hamid Nawab,** "Signals and Systems" Pearson Education Asia / PHI, 2nd edition, 1997. Indian Reprint 2002

2. **H. P Hsu, R. Ranjan,** "Signals and Systems", Scham's outlines, TMH, 2006

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UNIT 4
Fourier representation for signals – 1

4.1 Introduction:

Fourier series has long provided one of the principal methods of analysis for mathematical physics, engineering, and signal processing. It has spurred generalizations and applications that continue to develop right up to the present. While the original theory of Fourier series applies to periodic functions occurring in wave motion, such as with light and sound, its generalizations often relate to wider settings, such as the time-frequency analysis underlying the recent theories of wavelet analysis and local trigonometric analysis.

- In 1807, Jean Baptiste Joseph Fourier Submitted a paper of using trigonometric series to represent "any" periodic signal.

- But Lagrange rejected it!

- In 1822, Fourier published a book "The Analytical Theory of Heat" Fourier's main contributions: Studied vibration, heat diffusion, etc. and found that a series of harmonically related sinusoids is useful in representing the temperature distribution through a body.

- He also claimed that "any" periodic signal could be represented by Fourier series. These arguments were still imprecise and it remained for P.L.Dirichlet in 1829 to provide precise conditions under which a periodic signal could be represented by a FS.

- He however obtained a representation for aperiodic signals i.e., Fourier integral or transform

- Fourier did not actually contribute to the mathematical theory of Fourier series.

- Hence out of this long history what emerged is a powerful and cohesive framework for the analysis of continuous- time and discrete-time signals and systems and an extraordinarily broad array of existing and potential application.

The Response of LTI Systems to Complex Exponentials:

We have seen in previous chapters how advantageous it is in LTI systems to represent signals as a linear combinations of basic signals having the following properties.

Key Properties: for Input to LTI System

1. To represent signals as linear combinations of basic signals.
2. Set of basic signals used to construct a broad class of signals.
3. The response of an LTI system to each signal should be simple enough in structure.
4. It then provides us with a convenient representation for the response of the system.
5. Response is then a linear combination of basic signal.

Eigenfunctions and Values :

- One of the reasons the Fourier series is so important is that it represents a signal in terms of eigenfunctions of LTI systems.

• When I put a complex exponential function like x(t) ejwt through a linear time-invariant system, the output is y(t) H(s)x(t) = H(s) ejwt where H(s) is a complex constant (it does not depend on time).

• The LTI system scales the complex exponential ejwt

Historical background

There are antecedents to the notion of Fourier series in the work of Euler and D. Bernoulli on vibrating strings, but the theory of Fourier series truly began with the profound work of Fourier on heat conduction at the beginning of the century. In [5], Fourier deals with the problem of describing the evolution of the temperature of a thin wire of length X. He proposed that the initial temperature could be expanded in a series of sine functions:

$$f(x) \quad b_n \sin nx \qquad (1)$$

$$\int_0 f(x) \sin nx \, dx. \qquad (2)$$

The Fourier sine series, defined in EiTs (1) and (2), is a special case of a more general concept: the Fourier series for a *periodic function*_ Periodic functions arise in the study of wave motion, when a basic waveform repeats itself periodically. Such periodic waveforms occur in musical tones, in the plane waves of electromagnetic vibrations, and in the vibration of strings_ These are just a few examples_ Periodic effects also arise in the motion of the planets, in ac-electricity, and (to a degree) in animal heartbeats_

A function f is said to have period F if $f + F) = f(x)$ for all x_ For notational simplicity, we shall trict our discussion to functions of period 27r There is no loss of generality in doing so, since we can always use a simple change of scale s — (Pi 27r)t to convert a function of period F into one of period 27r_

If the function f has period 2-7r, then its *Fourier series* is

$$c_0 + \sum^{\infty} \{a_n \cos nx + b_n \sin nx\} \qquad (4)$$

with *Fourier coefficients* c_o, a_n, and b_i, defined by the integrals

$$c_0 = \frac{1}{4} \int f(x)\, dx \qquad (5)$$

$$a_n = \frac{-}{\pi} \int_{-}^{\pi} f(x) \cos nx \, dx,$$

$$b_n = \frac{1}{-} \int f(x) \sin nx \, dx. \qquad (6)$$

The following relationships can be readily established, and will be used in subsequent sections for derivation of useful formulas for the unknown Fourier coefficients, in both time and frequency domains.

$$\int_0^T \sin(kw_o t)dt = \int_0^T cos(kw_o t)dt = 0 \qquad (1)$$

$$\int_0^T \sin^2(kW_o t)dt = \int_0^T \cos^2(kW_o t)dt = \frac{T}{2} \qquad (2)$$

$$\int_0^T COS(kW_o t)\sin(gw_o t)dt = 0 \qquad (3)$$

$$\int \sin(kw_o t)\sin(gw_o t)dt = 0 \qquad (4)$$

$$\int_0 \cos(kw_o t)\cos(gw_o t)dt = 0 \qquad (5)$$

where

$$w0 \quad 2,f \qquad (6)$$

$$f = \frac{}{T} \qquad (7)$$

where f and T represents the frequency (in cycles/time) and period (in seconds) respectively. Also, k and g are integers.

A periodic function $f(t)$ with a period T should satisfy the following equation f

$$(t + T) = f(t) \qquad (8)$$

Example 1

Prove that

$$\int_0^T \sin(kw_o t) = 0$$

for

$$w_o \quad 2\#'$$

$$f = \frac{}{T}$$

and k is an integer.

Solution

Let

$$A = \int_0^T \sin(kw_o t)dt \qquad (9)$$

$$= [\overline{kw}\,) \mathbf{\cos(kw_o t)}]$$

$$A = [\overline{kw}\,)jcc\ os(kw_o\ T) — cos(0)]1 \qquad (10)$$

$$= [\overline{kw}\,j\cos(k27r) — 1]$$

$$=0$$

Example 2

Prove that

$$f \sin^2(kw_o t) \quad = \frac{7}{2}$$
0

for

$w_o \quad 27tf$

$f = \frac{—}{T}$

and k is an integer.

Solution

Let

$$\underline{B =} \; f \sin^2(kw_o t)dt \qquad (11)$$
with limits 0 to T

Recall

$$\sin^e (a) \quad = \frac{1 — \cos(2,a)}{2} \qquad (12)$$

Thus,

$$\underline{B =} \; L—2 — \frac{—1}{2} \cos(2kwot|^1 t \qquad (13)$$

[L2 j ' —22))sin(2kwot) | T 1 Jo

$$B \quad [2— \frac{1}{4kwo} \sin(2kw_o T)1 —| rL01i \qquad (14)$$

$\frac{T}{2} \quad [4kw)sin(2k \quad *27c)$

$\frac{T}{2}$

Example 3

Prove that

$$1 \sin(gw_o t)\cos(kw_o t) \quad \underline{= 0}$$
0

for

$w_o \quad 27tf$

$f = \frac{—}{T}$

and k and g are integers.

Solution

Let

$$C = \; 1 \sin(gw_o t)\cos(kw_o t)dt \qquad (15)$$
with limits 0 to T

Recall that

$$\sin(a + /3) = \sin(a)\cos(i3)+ \sin(fl)\cos(a) \quad (16)$$

Hence,

$$C= \int_0^T f[\sin[(g + k)w_o t] — \sin(kw_o t)\, cos(gw_o t)kit \quad (17)$$

$$\mathbf{f}_0^T \sin[(g + k)w_o tkit — \mathbf{f}_0^T \sin(kw_o t)\cos(gw_o t)dt \quad (18)$$

From Equation (1),

$$1_0^T [\sin(g + k)w_o t] dt = 0$$

then

$$C=0— i_0^T \sin(kw_o t)\, cos(gw_o t)dt \quad (19)$$

Adding Equations (15), (19), $2C = \mathbf{f}_0^T \sin(gw_o t) \cos(kw_o t)dt — \mathbf{f}_0^T \sin(kw_o t)\cos(gw_o t)dt$

$$= \mathbf{f}_0^T sitagw_o t)— (kw_o t)kit = 1_0^T \sin[(g — k)w_o dd\, t \quad (20)$$

2C = 0 , since the right side of the above equation is zero (see Equation 1). Thus,

$$C= 1_0^T \sin(gw_o t)\cos(kw_o t)dt = \quad 0 \quad (21)$$

$\underline{=0}$

Example 4

Prove that

$$1_0^T \sin(kw_o t)\sin(gw_o\, t)dt = 0$$

for

$w_o \quad 2\#'$

$f = \dfrac{}{T}$

k, g = integers

Solution

$$\text{Let } D = 1_0^T \sin(kw_o t)\sin(gw_o\, t)dt \quad (22)$$

Since

$$\cos(a +/3) = \cos(a)\cos(i3)— \sin(a)\sin(fl)$$

or

$$\sin(a)\sin(/3)= \cos(a)\cos(3)— \cos(a +13)$$

Thus,

$$\underline{D=} \, 1_0^T c\, os(kw_o t)\, c\, os(gw_o\, t)dt — \mathbf{f}_0^T co4(k + g)wA\, lt \quad (23)$$

From Equation (1)

$$\int_0^T \cos((k-g)w_0 t)\,dt = 0$$

then

$$D = \int_0^T \cos(kw_0t)\cos(gw_0t)dt = 0 \qquad (24)$$

Adding Equations (23), (26)

$$2D = \int_0^T \sin(kw_0t)\sin(gw_0t) + \int_0^T \cos(kw_0t)\cos(gw_0t)dt$$

$$\int \cos(kw_0t - gw_0t)dt \qquad (25)$$

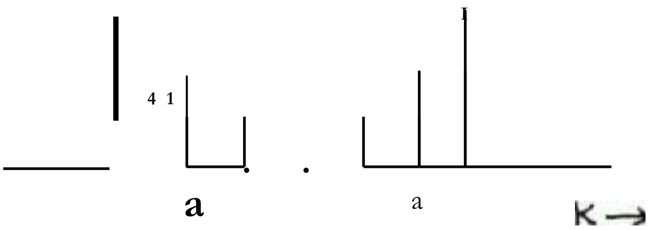
$$\int_0^T \cos((k-g)w_0 t)dt$$

2D = 0, since the right side of the above equation is zero (see Equation 1). Thus,

$$D = \int_0^T \sin(kw_0t)\sin(gw_0t)dt = 0 \qquad (26)$$

Recommended Questions

1. Find x(t) if the Fourier series coefficients are shown in fig. The phase spectrum is a null spectrum.



2. Determine the Fourier series of the signal x(t)=3 Cos(nt/2 + n/3). Plot the magnitude and phase spectra.

3. Show that if x[n] is even and real. Its Fourier coefficients are real. Hence fins the DTFS of the signal tii,i Efiln — 2p j

4. State the condition for the Fourier series to exist. Also prove the convergence condition. [Absolute integrability].

5. Prove the following properties of Fourier series. i) Convolution property ii) Parsevals relationship.

6. Find the DTFS harmonic function of x(n) = A Cos (27m/No). Plot the magnitude and phase spectra.

7. Determine the complex Fourier coefficients for the signal.
X(t)= {t+1 for -1 < t< 0; 1-t for 0 < t < 1 which repeats periodically with T=2 units. Plot the amplitude and phase spectra of the signal.

8. State and prove the following of Fourier transform. i) Time shifting property ii) Time differentiation property iii) Parseval's theorem.

UNIT 5: Fourier representation for signals — 2 **Teaching hours: 6**

Fourier representation for signals — 2: Discrete and continuous Fourier transforms(derivations of transforms are excluded) and their properties.

TEXT BOOK

Simon Haykin and Barry Van Veen "Signals and Systems", John Wiley & Sons, 2001.Reprint 2002

REFERENCE BOOKS

1. Alan V Oppenheim, Alan S, Willsky and A Hamid Nawab, "Signals and Systems" Pearson Education Asia / PHI, 2nd edition, 1997. Indian Reprint 2002

2. **H. P Hsu, R. Ranjan,** "Signals and Systems", Scham's outlines, TMH, 2006

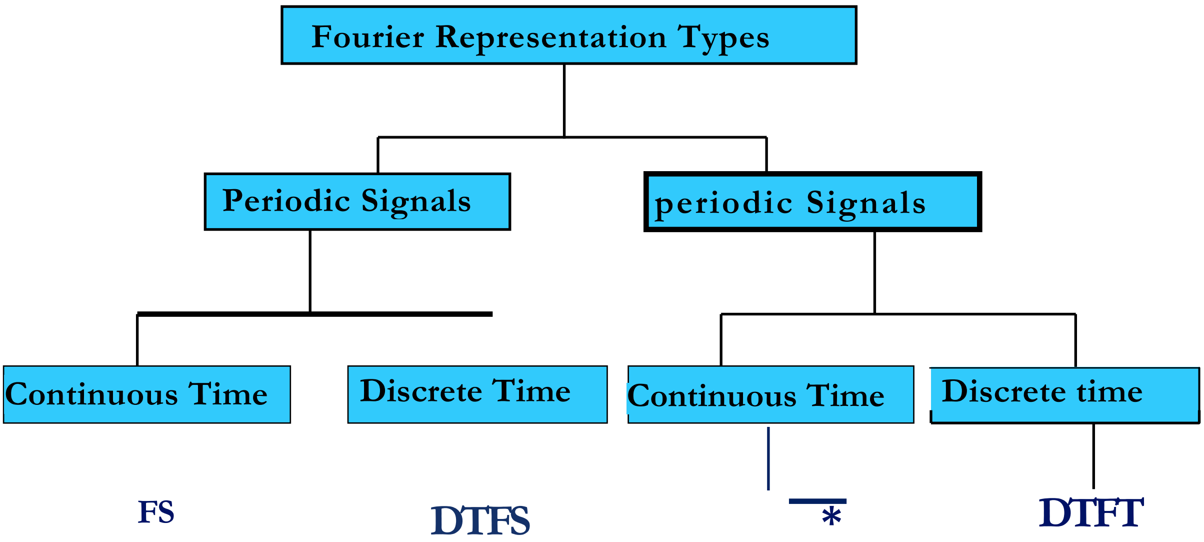
3. **B. P. Lathi,** "Linear Systems and Signals", Oxford University Press, 2005

4. Ganesh Rao and Satish Tunga, "Signals and Systems", Sanguine Technical Publishers, 2004

UNIT 5
Fourier representation for signals — 2

5.1 Introduction:

Fourier Representation for four Signal Classes



5.2 The Fourier transform

5.2.1 From Discrete Fourier Series to Fourier Transform:
Let x [n] **be a nonperiodic sequence** of finite duration. That is, for some positive integer *N*,

$$x[n] = 0 \qquad |n| > N_1$$

Such a sequence is shown in Fig. *6-1(a).* Let *x,Jn]* be a periodic sequence formed by repeating x [n] with fundamental period *No* as shown in Fig. 6-1(b). If w, met have

$$\lim_{N_0 \to \infty} x_{N_0}[n] = x[n]$$

The discrete Fourier series of *xNo[n]* is given *by*

$$x_{N_0}[n] = \sum_{k=\langle N_0 \rangle} c_k e^{jk\Omega_0 n} \qquad \overline{N}$$

$$e_k \quad \frac{1}{N_0} \sum_{n = <N_0>} x_{N_0}[n] \; e^{-jk\Omega_0 n}$$

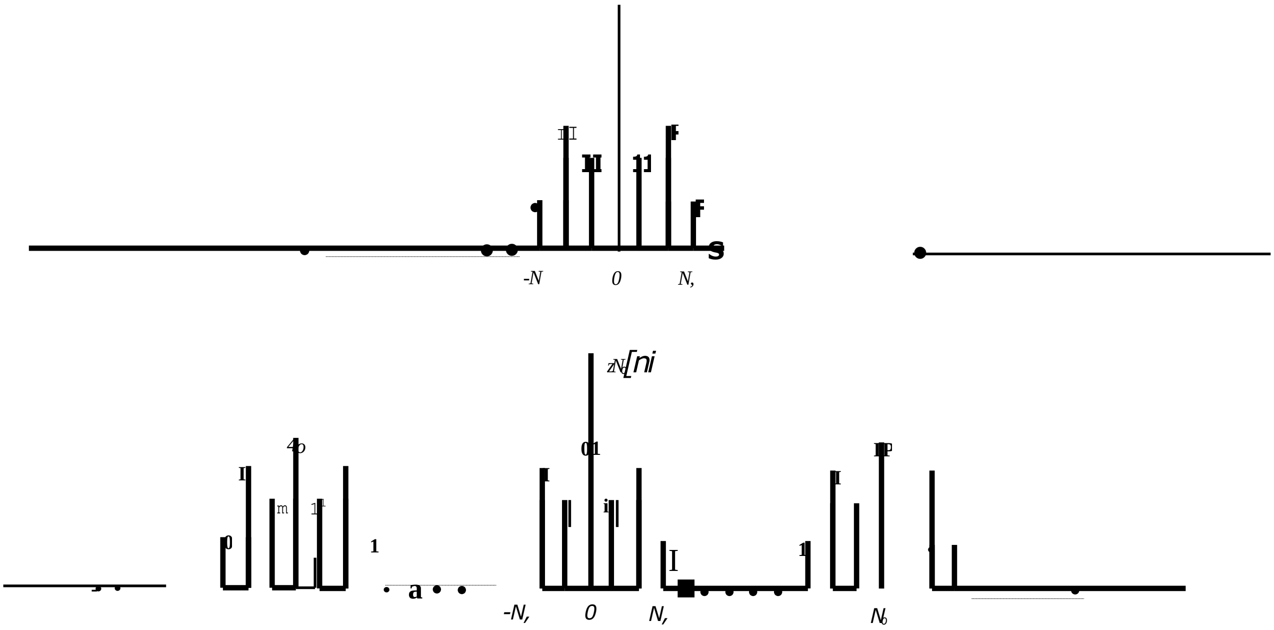


Fig. 6-1 (a) Nonperiodic finite sequence *4n]; (b)* periodic sequence formed by periodic extension of xrn].

$$c_k = \frac{1}{N_0} \sum_{n=-N_1}^{N_1} x[n] e^{-jk\Omega_0 n} = \frac{1}{N_0} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\Omega_0 n}$$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

the Fourier coefficients c_k can be expressed as

$$= \frac{1}{N_0} X(k\Omega_0)$$

$$x_{N_0}[n] = \sum_{k=<N>} \frac{1}{N_0} X(k\Omega_0) e^{jk\Omega_0 n}$$

$$x_{N_0}[n] = \frac{1}{2\pi} \sum_{k=<N_0>} X(k\Omega_0) \; e^{jk\Omega_0 n} \Omega_0$$

Properties of the Fourier transform

__Periodicity__

As a consequence of Eq. (6.41), in the discrete-time case we have to consider values of R(radians) only over the range() < f2 < 27(or $-\pi < \Omega < \pi$, while in the continuous-time case we have to consider values of ω (radians/second) over the entire range $-\infty < \omega < \infty$.

$$X(\Omega + 2\pi) = X(\Omega)$$

__Linearity:__

$$a_1 x_1[n] + a_2 x_2[n] \leftrightarrow a_1 X_1(\Omega) + a_2 X_2(\Omega)$$

Time Shifting:

$\sum_{n}$ — n_o 1 1)

Frequency Shifting:

$$e^{j\Omega_0 n}x[n] \leftrightarrow X(\Omega - \text{no})$$

Conjugation:

x xinj ..(--'X*(— Ω)

Time Reversal:

(-i (-n)

Time Scaling:

$$x(at) \leftrightarrow \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$$

Duality:

The duality property of a continuous-time Fourier transform is expressed as

X(/)'•—:27i-x(—to)

There is no discrete-time counterpart of this property. However, there is a duality between the discrete-time Fourier transform and the continuous-time Fourier series. Let

tn1 .V i ! 2)

$$X(\Omega) \quad \sum_{=-} x[n]e^{-j\Omega n}$$

X(11+ 27r) = X(Si)

Since 11 is a continuous variable, letting CI = t and n—k

$$\text{"t(t)} \quad \sum x[-k]e^{jkt}$$

Since X(t) is periodic with period To = 2 it and the fundamental frequency co_o = 27r/T_o = 1 , Equation indicates that the Fourier series coefficients of X(t) will be x [- k] . This duality relationship is denoted by

X(I) a c_k =

where FS denotes the Fourier series and c, are its Fourier coefficients.

Differentiation in Frequency:

$$[nI \longrightarrow j \frac{dX(.U)}{dS}$$

Differencing:

i % $1] \longleftrightarrow (1 = e^{-'''})X(0)$

The sequence x[n] -x[n — 1] is called the first difference sequence. Equation is easily obtained from the linearity property and the time-shifting property

Accumulation:

$$\sum_{k=-}x[k] \; 4 \; X(0)5(0) \pm \frac{1}{1 - e^{-j\Omega}} X(\Omega) \qquad -\sim-$$

Note that accumulation is the discrete-time counterpart of integration. The impulse term on the right-hand side of Eq. ***(6.57)*** reflects the dc or average value that can result from the accumulation.

Convolution:

$$x_1[n] * x_2[n] \longleftrightarrow X_1(\Omega)X_2(\Omega)$$

As in the case of the z-transform, this convolution property plays an important role in the study of discrete-time LTI systems.

Multiplication:

$$\text{fiirj}\ x_2[n] \longleftrightarrow \frac{}{2\pi} X_1(\Omega) \otimes X_2(\Omega)$$

where @ denotes the periodic convolution defined by

$$X_1(\Omega) \otimes X_2(\Omega) = 1\ X_1(0\ iX._2 \text{“}/ — 0)\ dO$$

The multiplication property (6.59) is the dual property of Eq. (6.58).

Parseval's Relations:

$$\sum_{-\infty} x_1[n]\, x_1[n] - \frac{1}{2\pi} \int_{2\pi} X_1(11)X_2(\quad -.1\cdot 1)\ \text{till}$$

$$\sum_{n=-\infty} |x[n]|^2 \quad \frac{}{2\pi}\int_{2\pi} |X(\Omega)|^2\, d\Omega$$

Summary

Prt criv	x(I),	X(160, Yaw)
Liiit a]	ux(E) + bjf]	caCro) t b Mu)
Time ShUMg-	x(t—En')	u¯i'üx(ioo)
Frequency Shifting	ei'""'x(t)	
Conjugation	s (0	
Time Reversal	x(—0	X(—jap)
Time and Fri:A tic Inc	x(at)	1 / |a|
Co nv citation	x(t) y(t)	X(fw)Y (iw)
Multi| lication	rtE)Y(t)	X(1rA) Y(jω)
I.Nfterentiatim	a	/4:.oXiijco)
[integration	i—	1 d'c0(0) + RX(0)6(ap)
Difit n ium Fie(Liency	ı	d / dω

Recommended Questions

1. Obtain the Fourier transform of the signal e $^{-at}$ u(t) and plot spectrum.

2. Determine the DTFT of unit step sequence x(n) = u(n) its magnitude and phase.

3. The system produces the output of yet) = e^{t} u(t), for an input of x(t) = e-2t.u(t). Determine impulse response and frequency response of the system.

4. The input and the output of a causal LTI system are related by differential equation

   $\frac{d\,2y\,(t)}{dt2}\;\frac{6dy\,(CI}{}\quad 8y(t)= 2x(t)$

   i) Find the impulse response of this system

   ii) What is the response of this system if x(t) = te^{at} u(t)?

5. Discuss the effects of a time shift and a frequency shift on the Fourier representation.

6. Use the equation describing the DTFT representation to determine the time-domain signals corresponding to the following DTFTs

   i) X(en= Cos(12)+j Sin(12)

   ii) X(ej^{a})={1, for ir/2<fl< it; 0 otherwise and X(ej^{a})=-4 **12**

7. Use the defining equation for the FT to evaluate the frequency-domain representations for the following signals:

   i) X(t)= e^{3t}u(t-1)

   ii) X(t)=e ^{t} Sketch the magnitude and phase spectra.

8. Show that the real and odd continuous time non periodic signal has purely imaginary Fourier transform. (4 Marks)

UNIT 6: Applications of Fourier representations **Teaching hours: 7**

Applications of Fourier representations: Introduction, Frequency response of LTI systems, Fourier transform representation of periodic signals, Fourier transform representation of discrete time signals.

TEXT BOOK

Simon Haykin and Barry Van Veen "Signals and Systems", John Wiley & Sons, 2001.Reprint 2002

REFERENCE BOOKS

1. **Alan V Oppenheim, Alan S, Willsky and A Hamid Nawab,** "Signals and Systems" Pearson Education Asia / PHI, 2nd edition, 1997. Indian Reprint 2002

2. **H. P Hsu, R. Ranjan,** "Signals and Systems", Scham's outlines, TMH, 2006

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4. **Ganesh Rao and Satish Tunga,** "Signals and Systems", Sanguine Technical Publishers, 2004

UNIT 6
Applications of Fourier representations

6.1 Introduction:

Fourier Series and LTI System

- Fourier series representation can be used to construct any periodic signals in discrete as well as continuous-time signals of practical importance.
  We have also seen the response of an LTI system to a linear combination of complex exponentials taking a simple form.
  Now, let us see how Fourier representation is used to analyze the response of LTI System.

Consider the CTFS synthesis equation for x(t) given by
Suppose we apply this signal as an input to an LTI System with impulse respose h(t). Then, since each of the complex exponentials in the expression is an eigen function of the **system. Then, with** *s k =jkwo* **' follows that the output is**

$$y(t) \quad \sum_{k=-\infty}^{+\infty} \text{RI } H(eiknejkwot$$

Thus y(t) is periodic with frequency as x(t). Further, if ak is the set of Fourier series coefficients for the input x(t), then { ($_e$'I ii s the set of coefficient for the y(t). Hence in LTI, modify each of the Fourier coefficient of the input by multiplying by the frequency response at the corresponding frequency

Example:

Consider a periodic signal x(t), with fundamental frequency 2n, that is expressed in the form

$$X(t) = \sum_{k=-3}^{+3} a_k e^{jk'} \quad \dots \quad (1)$$

where, ao.i. $ai=a-i= 1/4$, $a2=a-2=1/2$, $a3=a-3=1/3$,

Suppose that the this periodic signal is input to an LTI system with impulse response To calculate the FS Coeff. Of ofp y(t), lets compute the frequency response.The impulse response is therefore,

$$H\,) \quad = \quad e\text{-}1`Yrthr \quad \equiv -\frac{1}{1 + ico} e - e^{-j\omega t}$$

and

$$H((.0) = \frac{1}{1 + /et)}$$

Y(t) at coo 27c . We obtain,

$$Y(t) - \sum_{k=-3}^{+3} bkeR2' \quad \text{with } bk = akfl\ Ok27\bar{0}, \text{ so that}$$

$$= \frac{1}{\_\ j2\ m} \quad b2 = \frac{1}{2 +} \frac{1}{j4m -} \quad b3 = \frac{1}{3} \frac{1}{\pm\ j60}$$

$$b\_i = \frac{1}{-j2\ 7r)} \quad b2 \quad \frac{1}{2} \frac{1}{U - 140} \quad b3 = \frac{1}{3} \frac{1}{-j6TE\ )}$$

$b. = 1$

The above o/p coefficients. Could be substituted in

$$y(t) = \sum_{k=-3}^{+3} bkefic^{2}Trt$$

Finding the Frequency Response

We can begin to take advantage of this way of finding the output for any input once we have Fl(co).

To find the frequency response H(o) for a system, we can:

1. Put the input x(t) = e 'er` into the system definition
2. Put in the corresponding output y(t) = El(c.o) n
3. Solve for the frequency response H(a)). (The terms depending on t will cancel.)

Example:

Consider a system **with impulse response**

$$h(t) = \begin{cases} - & \text{for } t \in [0,5] \\ 0 & \text{otherwise} \end{cases}$$

Find the output correspondingto the input x(t) cos(1O t).

$$y(t) = \int_{-E} h(r)\, x(t - E)\, dr = \int_{\tau=0}^{5} \frac{1}{5} \cos(100 - 0)\, dt$$

$$y(t) - \left(-\frac{1}{10} \sin(10(t - \tau))\right)\Bigg|^{5} = \frac{1}{50}\, i(\sin(10t) - \sin(10(t-5)))01$$

Differential and Difference Equation Descriptions

Frequency Response is the system's steady state response to a sinusoid. In contrast to **differential** and difference-equation descriptions for a system, the frequency response description cannot represent initial conditions, it can only describe a system in a steady state condition. The differential-equation representation for a continuous-time system is

$$a_k \frac{d^k}{dt^k} y(t) \qquad \sum_{k=0}^{N} b_k \frac{d^k}{dt^k} x(0$$

$$\text{since, } \frac{d}{dt} g(t) \xleftarrow{FT} j\omega G(j\omega)$$

Rearranging the equation we get

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{2 \quad _0\ bk\ (itO^k}{\text{Ek—O}\ ^{ak}\ U^C O^k}$$

The frequency of the response is

$$(N) - \frac{}{X\ (j(0)} \quad \frac{bk\ Uw)^k}{EZ_{=0}\ ak\ (jco)^k}$$

Hence, the equation implies the frequency response of a system described by a linear constant-coefficient differential equation is a ratio of polynomials in j

The difference equation representation for a discrete-time system is of the form.

$$\sum_{k\ 0}^{N} aky[n - \quad = \quad \sum_{k\ o}^{M} bkx[n - k]$$

Take the DTFT of both sides of this equation, using the time-shift property.

$$- k] \xrightarrow{DTFT} - \sim G e J \sim$$

To obtain

$$\sum_{k\ 0}^{N} ak \left(e^{-j\omega}\right)^k Y\left(e^{j\omega}\right) \quad \sum_{k\ o}^{N} (e^{-ic} \quad k\ X(e^{jw})$$

Rewrite this equation as the ratio

$$\frac{Y(ej^w)}{X(ejw)} - \frac{2^1{}_{-0}\ bk\ (e^{il^k}}{EZ_{=o}\ ak\ (ejw)k}$$

ei6j

The frequency response is the polynomial in

$$11(e^{-16)}) = \frac{Y(eJ^W)}{} \quad \frac{EtL_o\ bk\ (ej^w)^k}{Ek=o\ ak\ (e^{jw})^k}$$

Differential Equation Descriptions

Ex: Solve the following differential Eqn using FT.

$$\frac{d^2}{dt2}y(t) + 4\ \frac{d}{c}y(t) + 5y(t) \quad 3\frac{d}{1t})c(t)+x(t)$$

For *all* t where, $x(t)$ **Soln** $(1 + e^{-t})u(t)$

:we have

$$\frac{d^2}{cit2}At) + 4 \frac{d}{dt}At) + 5y(t) \quad 3\frac{d}{cit}x(t) + x(t)$$

FT gives,

$$[Ci + 400 + 5]\ Y\ CIO \quad (3jo + 1)X(l6))$$

$$and\ x(t) \quad (1 + e^{-t})u(t) \qquad x(t) \quad u(t) + (e^{-t})u(t)$$

$$X(1(6) \quad \left(\frac{1}{j\omega} + ITS(a))\right) + \frac{1}{uw \_\_\_} + since \quad \xrightarrow{FT} \quad 7r(5(6.\_)) + \frac{1}{j)}$$

$$and (e^{-t})u(t) \xrightarrow{Fr} \frac{1}{jw + 1}$$

$$Ala)) = \frac{}{j} + ve5(w) + \frac{1}{(160+1)}$$

Hence we have

$$And \ Rfo))^2 + 4(*) + 5] \ Kiri)) \qquad (3joi + 1)X(jco)$$

1e

$$Y(j\omega) = \frac{(3jo) + 1)}{[00^2 + 406)) + 5]} X(j6))$$

$$YOco) = \frac{(3j\,to + \quad 1}{„RA-) + 2)^2 + 1] \quad ita)} + 716(6.0 + \frac{1}{(jti) + \quad 1]}$$

$$Y\ ) = \frac{(3jW + 1)}{Rico)^2 + 40,(0) + 5]}\left[\left(\frac{1}{j\omega} + TTS(CO)\right) \pm \frac{1}{+0}\right]$$

$$\boxed{r(joi) \qquad Y(1) + Y(2) + Y(3)}$$

$$Y(1\,6)) = \frac{(3\,jco + 1)}{[U(z) \quad 2)^2 + lliCd} + \frac{\mathit{TE}}{5}\delta(\omega) + \frac{(3j\ w + 1)}{[(fa) + 2)^2 + 1](ja) + 1)}$$

$$YUa) = \frac{(3jca + 1)}{[Uct) + 2)^2 + *co} + \frac{(3/(0,) = 0) + 1)45\,(0) = 1]}{[(j(\omega - 0) + 2)2 + 'Kw - 0)} + \frac{(3j4) + 1)}{Kied + 2)2 + li(ja) + 1)}$$

$$1^1(1) = \frac{(3iw + 1)}{Rico + 2)^2 + 11/to} Y(1) = \frac{A}{j\omega} + \frac{Bjw + C}{[[Ow + 2)^2 + 1]}$$

Performing partial fraction we get $A = \frac{1}{5}$, B $, C = \frac{11}{5}$

$$\boxed{Y(1) = \frac{1/5}{ja)} + \frac{-1/5jw + 11/5}{[(jw + 2)^2 + 1]}}$$

Similarly

$$Y(3) \quad \frac{(3jco + 1)}{[(jo) + 2)^2 + 1](ico + 1)}$$

$$Y(3) \quad \frac{R}{(ja) + 1)} \quad \frac{Pjco + Q}{[0co + 2)^2 + 1]}$$

Performing partial fraction we get $R \quad -1, P \quad 1, Q \quad 6$

$$\boxed{Y(3) = \frac{-1}{(jco+1)} + \frac{j\omega+6}{[(joi+2)^2+1]}}$$

$$Y(3) = \frac{-1}{Ow+1)} + \frac{ja) \pm 6}{[Ow+2)^2+1]\ YUc.o)} \qquad Y(1) + Y(2) + Y(3)$$

Hence, we have

$$Y(I) - \frac{\text{---}}{/co} + \frac{115\, jai + 11/5}{[Ow+2)^2+1]}$$

$$Y(2) \qquad \delta(\omega)$$

Readjusting

$$M\ ) = \frac{1/5}{j} + \frac{-1/5jw + 11/5}{[(f+2)^2+1]} + \frac{Tr}{5} 6(w) + \frac{-1}{+1)} + \frac{jto+6}{[(1w+2)^2+1]}$$

$$\boxed{Y(Ia)) = \frac{1}{jai} + \pi\delta(\omega)\ \frac{1}{U(4)+1)} + \frac{4j\,co + 41}{[\![\omega+2)^2+1]\!]}}$$

$$Y(/(o)) = \frac{}{Pz)} + \frac{\bar{n}}{5}\delta(\omega) + \frac{11/5 - 1/5\,jai}{[(ja)+2)^2+1]} + \frac{jw+6}{[(f+2)^2+1]}\ \frac{1}{(jco+1)}$$

we know that,

$$e^{-P^t} Cos\ oh,\ \textbf{WO}\ \xleftrightarrow{Fr} \frac{+fto}{[(p+{}_{/w})2 + {}_{(00}9}$$

$$e^{-P^t} \sin co.\ tu(t)\ \xleftrightarrow{Fr} \frac{(\omega}{[(i^3 + /0^2 + {}^{(0}0^2]}$$

Readjusting the last term, we get

$$Y(j\omega) = \frac{1}{5}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] \frac{1}{(jco+} + \frac{4}{} \frac{jc.o-1-2}{[fico\ I\ 2)^2+1]} + \frac{33}{5} \frac{1}{L[(jco+2)^2+i]]}$$

Now, taking the inverse Fourier Transform, we get

$$\boxed{y(t) = \frac{-}{5}140 - e^{-t}u(t) + \frac{-}{5}e^{-2t} \cos t\, u(t) + \frac{--}{5}e^{-2t} \sin t\, u(t)}$$

Differential Equation Descriptions

Ex: Find the frequency response and impulse response of the system described by the differential equation.

$$\frac{d^2}{dt^2}y(t) + 3\frac{d}{Tit}y(t) + 2y(t) \qquad 2\frac{d}{dt}x(t) + x(t)$$

Here we have N=2, M=1. Substituting the coefficients of this differential equation in

$$Huw) - \frac{\quad}{X0^{\cdot}60)} \quad \frac{E \ {}_{=0} b_k (j\omega)^k}{E_{k.} \ {}_0 ak(r,o)k}$$

<u>Differential Equation Descriptions</u>

We obtain

$$Wico) \quad \frac{2j6) + 1}{U(0)^2 + 3.itt) + 2}$$

The impulse response is given by the inverse FT of H(jco). Rewrite HO co) using the partial fraction expansion.

$$1\text{-}1U(0) = \frac{A}{jco + 1} + \frac{B}{jto\ \text{-}I\text{-}\ 2}$$

Solving for A and B we get, A=-1 and B=3. Hence

$$1/(joi) \quad \frac{-1}{\_Rd + 1} + \frac{3}{jto + 2}$$

The inverse FT gives the impulse response

$$\boxed{3e^{-2t} \quad u(t) - e^{-t} u(t)}$$

<u>Difference Equation</u>

Ex: Consider an **LTI** system characterized by the following second order **linear** constant coefficient erence equation.

$$y[n] \quad 1.3433An - 1] - 0.9025y[n - 2] + x[n]$$
$$- 1.41424n - 1] + x[n - 2]$$

Find the frequency response of the system.

Soin:

$$y[n] \quad 1.3433y[n - 1] - 0.9025y[n - 2] + x[n]$$
$$- 1.4142x[n - 1] + x[n - 2]$$

$$Y(ejl \qquad (ejl$$
$$- 0.9025(e^{-j2\omega})Y(e^{j\omega}) + X(e^{j\omega})$$
$$- 1.4142 \quad K \quad {}^{j\omega}) + (e^{-j2\omega})X(e^{j\omega})$$

$$we\ know, yin - \xleftarrow{DIFI} e^{-jK}w \ Y(ei')$$

$$= \frac{\quad}{X(e-w)}$$

$$= \frac{1 - \quad + \quad e^{-\}2} G ` '}{1 - \quad 1.3433e^{-Pi'} + 0.9025e^{-i^2 ü)}}$$

Ex: If the unit impulse response of an LTI System is h(n)=anu[n], find the response of the system to an input defined by $x[n] = ,8^{nu} if,^1n^1-]$ where p, a < 1 *and* a # 13

Solo:

$Y[n] = h[n] * x[n]$

Taking DTFT on both sides of the equation, we get

$$1/(ei') = 11(ei')X(e1') \qquad Y(eil \quad \frac{1}{1 - ae^{-iw}} \cdot \frac{1}{1 - 13e^{-i°)}}$$

$$Y(ejw) \quad \frac{1}{1 - cte\text{-}i6J} \times \frac{1}{1 - fie^{-lc)}} = \frac{A}{1 - ae^{-/U'}} x \frac{B}{1 - 13e^{-iw}}$$

where A and B are constants to he found by using partial fractions

$$Let,\ e^{-iw} \equiv v \qquad Then,\ Y(e^{-j'}) = \frac{A}{1 - av} \times \frac{B}{1 - \beta v}$$

By performing partial fractions, we get $A = \frac{a}{a -}$ $\underline{B=} \frac{-f3}{a - p}$

$$There\ f\ ore\ ,Y(e\ jw) \quad \frac{\frac{a}{a -}}{1 - \ a e c')} \times \frac{\frac{-/3}{a - f3}}{1 - \ /;e\text{-}\sim''}$$

Taking inverse DTFT, we get

$$\boxed{y[n] \quad \frac{a}{a - fl} \quad - \frac{\quad}{a - ig} \ uni \ u[n]}$$

Sampling

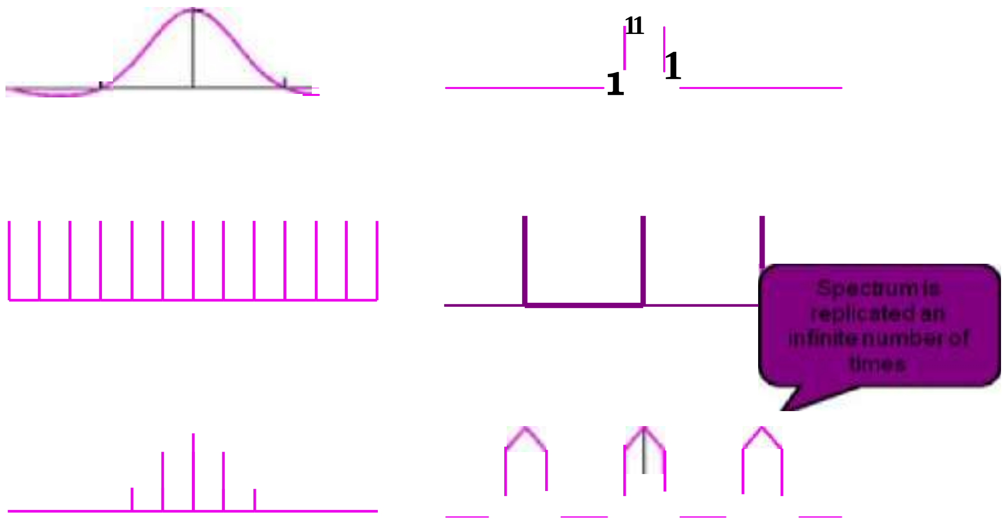
In this chapter let us understand the meaning of sampling and which are the different methods of sampling. There are the two types. <u>Sampling Continuous-time signals</u> and **Sub-sampling.** In this again we have *Sampling Discrete-time signals.*

Sampling Continuous-time signals

Sampling of continuous-time signals is performed to process the signal using digital processors. The sampling operation generates a discrete-time signal from a continuous-time signal.DTFT is used to analyze the effects of uniformly sampling a signal.Let us see, how a DTFT of a sampled signal is related to FT of the continuous-time signal.

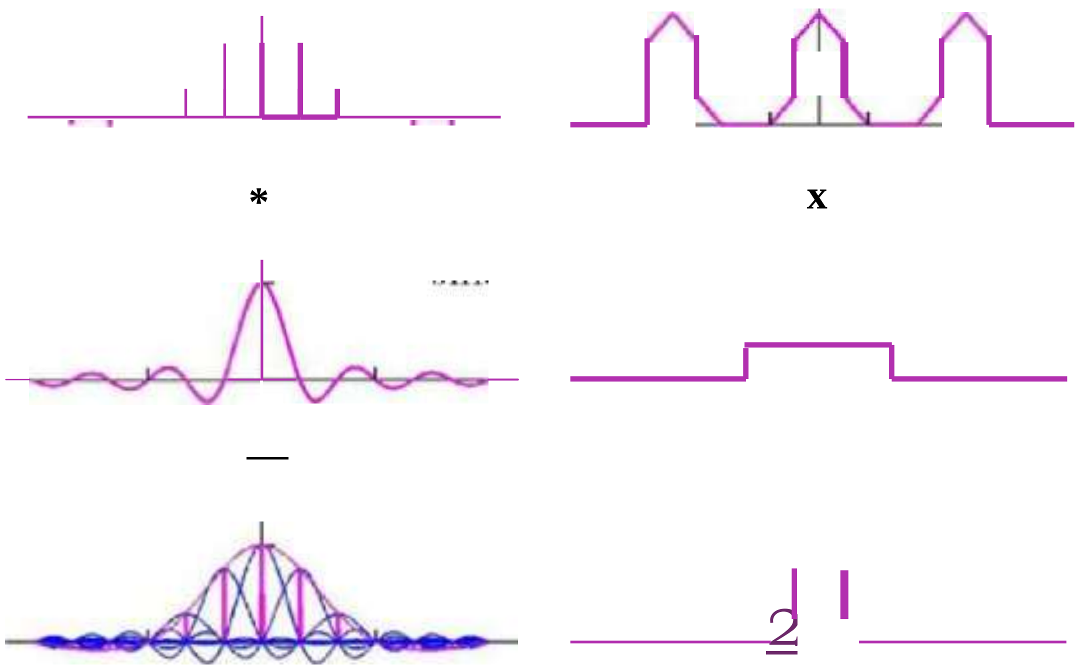
- Sampling: Spatial Domain: A continuous signal x(t) is measured at fixed instances spaced apart by an interval ^{C}T'. The data points so obtained form a discrete signal x[n]=x[nT]. Here, AT is the sampling period and 1/ AT is the sampling frequency.Hence, sampling is the multiplication of the signal with an impulse signal.

Sampling theory



Rjco)

Reconstruction theory



x(t)

FON)

Samplinu: Spatial Domain

From the Figure we can see

Where x[n] is equal to the samples of x(t) at integer multiples of a sampling **interval T**

$$xs(t) \quad \sum_{=}^{+\infty} x(n)\, o(t - nr)$$

Now substitute x(nT) for x[n]to *obtain .*

$$xs(t) \quad \sum_{n=-\infty}^{+\infty} x(rre)\, 8(t - nc)$$

since x(t)5(t — nr) x(nr)5(t nr)

we may ***rewrite xs(t) as a product of time functions***

$$xs(t) \quad x(t)p(t) \qquad \textit{where,} \quad p(t) \qquad - nr)$$

Hence, Sampling is the multiplication of the signal with an impulse train.

The effect of sampling is determined by relating the FT **of** Xs (t) **to** the FT of X(t) Since Multiplication in the time domain corresponds to convolution in the frequency domain, we have

$$Xs(j\,\&.)) = \frac{1}{2ff,} \quad X\,W$$

Substituting the value of *P(i 6a)* as the FT of the pulse train i.e

$$p(t) = \sum_{n=-\infty} (5(t - nT)$$

We get,

$$P(jw) = \frac{}{T} \sum_{41=-\infty}^{+02.} \delta(\omega - k\omega s)$$

where, cos $= \frac{}{T}$; *is* ***the sampling frequency. Now***

$$XsUco) - \frac{1}{-ff} X(w) * - \sum_{n=-\infty}^{+00} \delta(\omega \quad k\omega s)$$

$$\boxed{s(j\omega) \quad \frac{1}{I} \sum_{n=-\infty}^{+\infty} X(j(\omega - k\ ))}$$

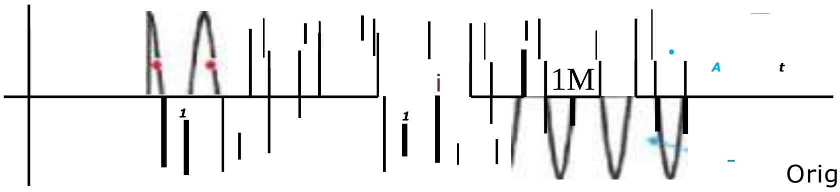
The FT of the sampled signal is **given by an infinite sum of shifted version of the original** signals FT and the offsets are integer multiples **of co$_s$_**

Aliasing : an example

Frequency of original signal is 0.5 oscillations per time unit). Sampling frequency is also 0.5 oscillations per time unit). Original signal cannot be recovered.

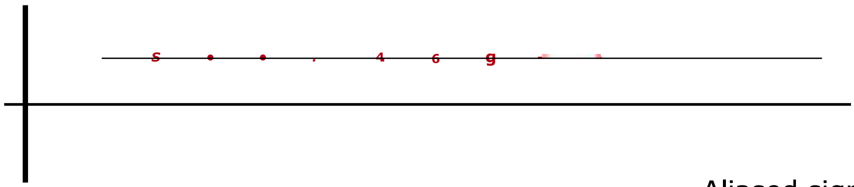
Aliasing Ex: 1

Sampling points x[n]



Original signal x(t)

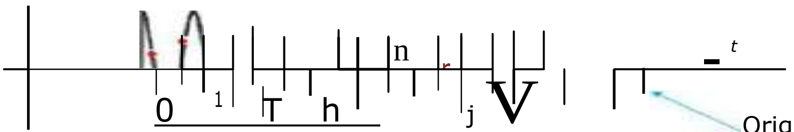
Sampling frequency ws =115cyclesiunit time



Aliased signal which is reconstructed

Aliasing Ex:2

Sampling points x[n]



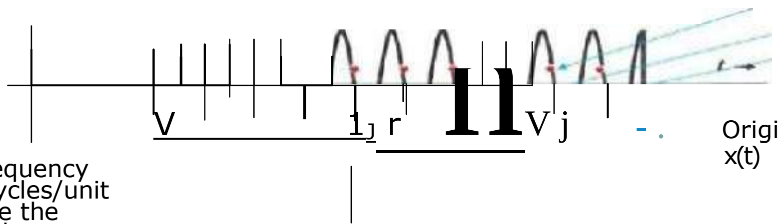
Original signal x(t)

Sampling frequency ws =0 7cyclestunit time

Aliased signal appear like a sirie wave but of lower frequency, original signal is lost

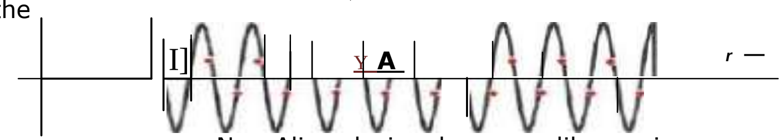
Non-Aliasing: Ex 3

Sampling points x[n]



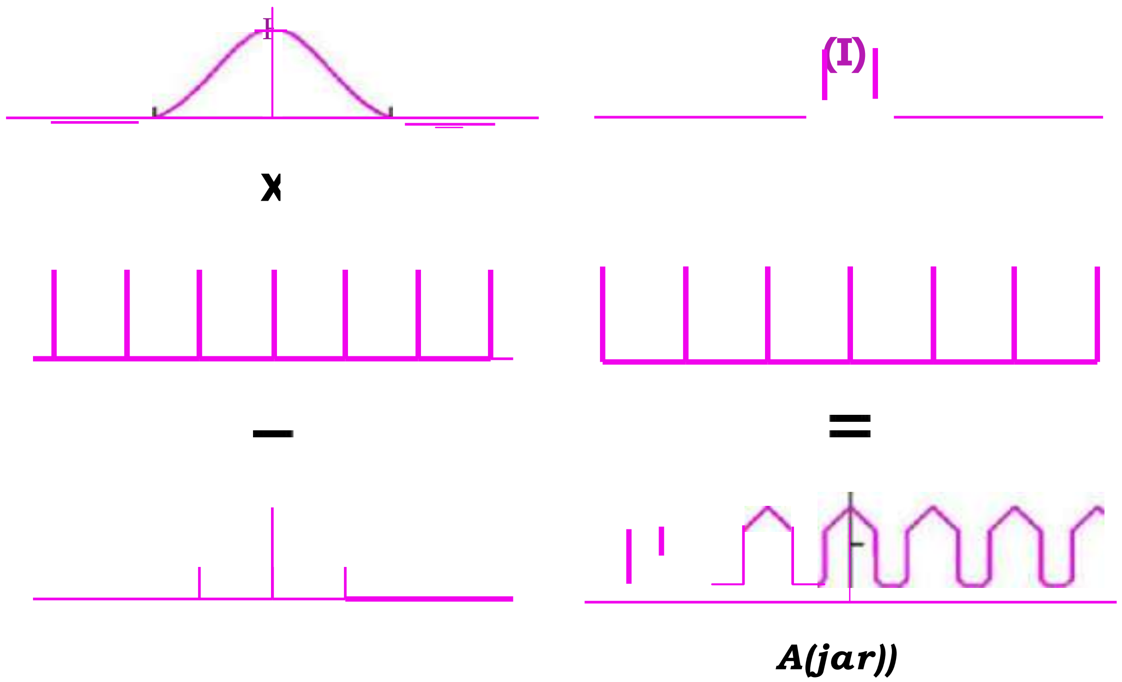
Origina l signal x(t)

Sampling frequency u..rs =1.0 cycles/unit time i.e twice the frequency of the input

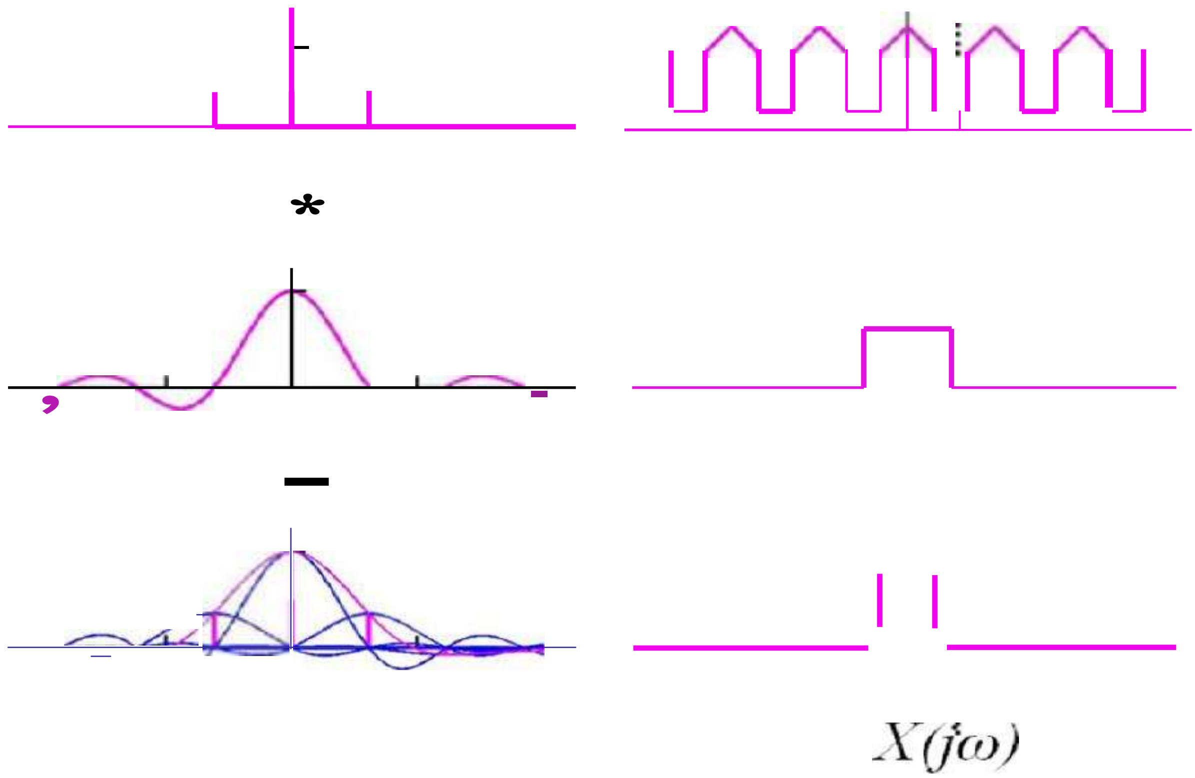


Non-Aliased signal appear like a sine wave but of lower frequency, original signal is lost

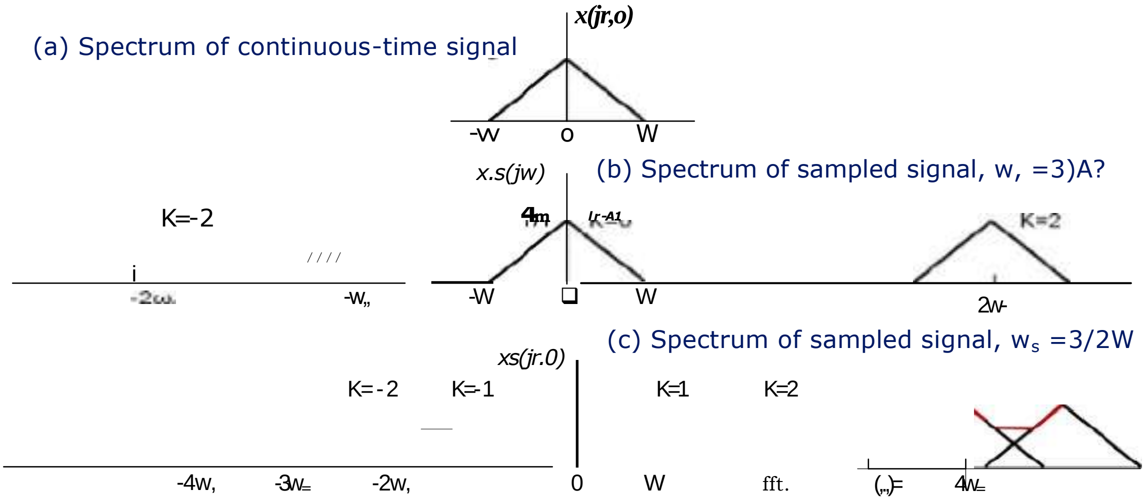
Sampling below the Nyquist rate



Reconstruction below the Nyquist rate



FT of sampled signal for different sampling frequency



Reconstruction problem is addressed as follows.
Aliasing is prevented by choosing the sampling interval T so that $\omega_s > 2W$, where W is the highest frequency component in the signal. This implies we must satisfy $T < \pi/W$.

Also, DTFT of the sampled signal is obtained from sOco using the relationship D.= coT, that is

$$x[n] \xleftrightarrow{DTFT} X(e^{j\omega}) = Xa(ico) \Big| = tu$$

This scaling of the independent variable implies that $\omega = \omega_s$ corresponds to $\Omega = 2\pi$

Subsampling: Sampling discrete-time signal

FT is also used in discrete sampling signal.

Let)'I = xrcul1 be a subsampled version x[n], where q is a positive integer.
Relating DTFT of y[n] to the DTFT of x[n], by using FT to represent x[n] as a sampled versioned of a continuous time signal x(t).
Expressing now y[n] as a sampled version of the sampled version of the same underlying CT x(t) obtained using a sampling interval q that associated with x[n]
We know to represent the sampling version of x[n] as the impulse sampled CT signal with sampling interval T.

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(n)\,\delta(t - nT)$$

Suppose, x[n] are the samples of a CT signal x(t), obtained at integer multiples of T. That is, x[n]=x[nT]. Let $x(t) \longleftrightarrow X(j\omega)$ and applying it to obtain

$$\boxed{X_\delta(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(j(\omega - k\omega_s))}$$

Since y[n] is formed using every qth sample of x[n], we may also express y[l] as a sampled version of x(t).we have *Yfrd* xrcicni x(nqT)

Hence, active sampling rate for An] is T'=qT. Hence

ya(t) *x(t)* ***8(t —*** $\xrightarrow{FT}$ ***Y6(j w)*** = $\frac{1}{}$ **1** ***X((co —icco2))***

n=—. ***k=—∞***

Hence substituting r=q T, and $cl)_s' = co_s/q$

$$N i C 0) = \frac{1}{q\tau} \sum_{k=-\infty}^{+\infty} X0 (Cti \quad Ws))$$

We have expressed both *Ys(i co) and Mitt))* as a function of

Expressing X(1° as a function of 5(itc1) . Let us write k/q as a proper function, we get

$$1 + \frac{m}{q}$$

where I is the integer portion of $\frac{k}{q}$, and m is the remainder

allowing k to range from —oo to + co corresponds

to having I range *from —as to + co and m from 0 to q — 1*

$$Y6(j(0) \quad \sum_{m=0}^{q} \frac{1}{T} \sum_{1=-00}^{+\infty} XS\left(j\left(\omega - l\omega_s - \frac{711}{q}\omega_s\right)\right)$$

$$Y_\delta(j\omega) \quad \frac{1}{q}\sum_{7n=0}^{q-1} X8\left(j\left(\omega \quad \frac{M}{} \quad co5\right)\right)$$

which represents a. sum o f shifted versions of

Ufa)) normalized by **q.**

Converting from the FT representation back to DTFT

and substituting S1 = cof above

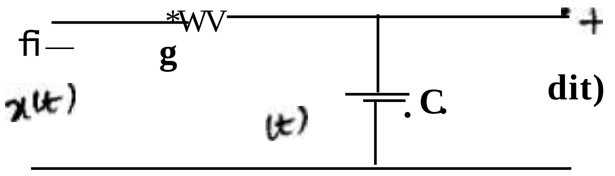
and also X(ei^n) X5(j f2/T) , we write this result as

$$\boxed{Yo(e^{j\Omega}) \quad \frac{1}{q}\sum_{7n=0}^{q-1} \quad c \quad i \quad e}$$

where, $X_q(e^{j\Omega}) = X(ei^n iq)$ — a scaled DTFT version

Recommended Questions

1. Find the frequency response of the RLC circuit shown in the figure. Also find the impulse response of the circuit



FIE.Q64 h)

2. The input and output of causal LTI system are described by the differential equation.

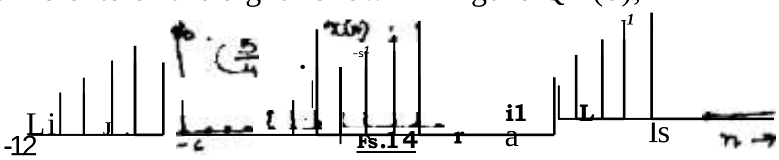
$$\frac{d'y(t)}{dt^2} + 3\frac{dY}{dt} + 2\,y(t) - x(t)$$

i) Find the frequency response of the system
ii) Find impulse response of the system
iii) What is the response of the system if $x(t) = te^{-t}u(t)$. (10 Marks)

3. If x(04-0C(f). Show that x(t)Coswot4-+1/2[X(f-fo)+X(f-fo)] where w0=27(fo

4. The input $x(t) = e^{-3t}u(t)$ when applied to a system, results in an output y (t) the e^{t} u(t). Find frequency response and impulse response of the system. **(07 Marks)**

5. Find the DTFS co-efficients of the signal shown in figure Q4 (b),



6. State sampling theorem. Explain sampling of continuous time signals with relevant expressions and figures.

7. Find the Nyquist rate for each of the following signals:
i) $x(t) = \text{sinc}(200t)$ ii) $x(t) = \text{sinc}^{e}(500t)$

UNIT 7: Z-Transforms —1 **Teaching hours: 7**

Z-Transforms — 1: Introduction, Z — transform, properties of ROC, properties of Z — transforms, inversion of Z — transforms.

TEXT BOOK

Simon Haykin and Barry Van Veen "Signals and Systems", John Wiley & Sons, 2001.Reprint 2002

REFERENCE BOOKS

1. Alan V Oppenheim, Alan S, Willsky and A Hamid Nawab, "Signals and Systems" Pearson Education Asia / PHI, 2nd edition, 1997. Indian Reprint 2002

2. **H. P Hsu, R. Ranjan,** "Signals and Systems", Scham's outlines, TMH, 2006

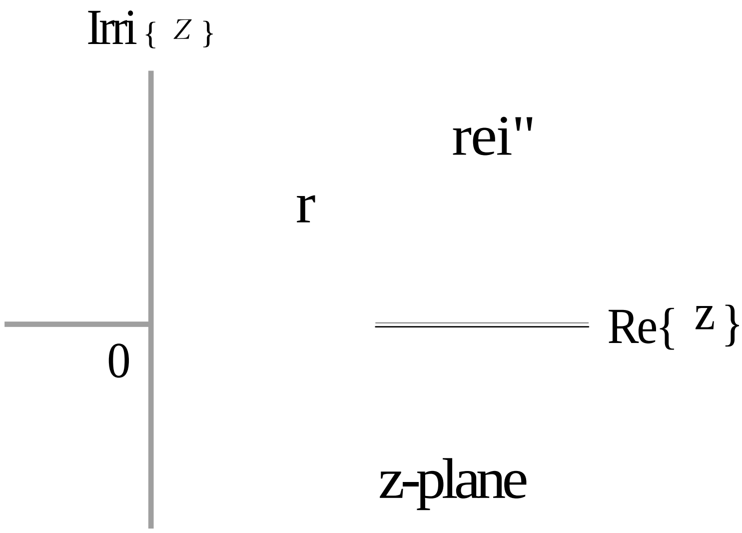
3. **B. P. Lathi,** "Linear Systems and Signals", Oxford University Press, 2005

4. Ganesh Rao and Satish Tunga, "Signals and Systems", Sanguine Technical Publishers, 2004

UNIT 7
Z-Transforms – 1

7.1 Introduction to z-transform:

The z-transform is a transform for sequences. Just like the Laplace transform takes a function of t and replaces it with another function of an auxiliary variable s. The z-transform takes a sequence and replaces it with a function of an auxiliary variable, z. The reason for doing this is that it makes difference equations easier to solve, again, this is very like what happens with the Laplace transform, where taking the Laplace transform makes it easier to solve differential equations. A difference equation is an equation which tells you what the k+2th term in a sequence is in terms of the k+1 th and kth terms, for example. Difference equations arise in numerical treatments of differential equations, in discrete time sampling and when studying systems that are intrinsically discrete, such as population models in ecology and epidemiology and mathematical modelling of mylinated nerves. Generalizes the complex sinusoidal representations of DTFT to more
generalized representation using complex exponential signals



- **It** is the discrete time counterpart of Laplace transform

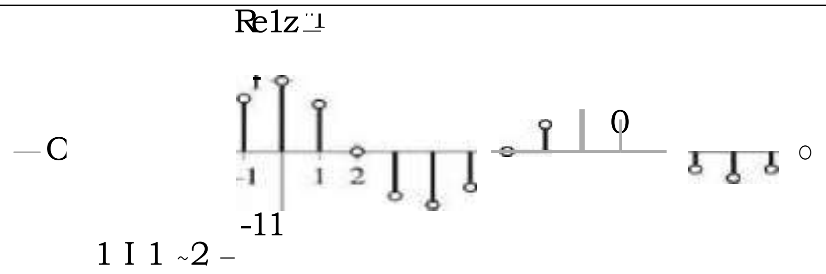
The z-Plane

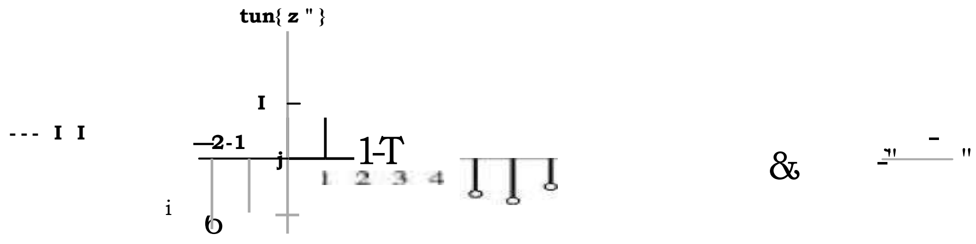
- Complex number $z = re^{j\theta}$ s represented as a location in a complex plane *(z plane)*

7.2 The z-transform:

- Let $z = re^{j\Omega}$ be a complex number with magnitude and angle
- The signal $x[n] = z^n$ is a complex exponential and $x[n] = r^n \cos(\Omega n) + jr^n \sin(\Omega n)$
- The real part of $x[n]$ is exponentially damped cosine
- The imaginary part of $x[n]$ is exponentially damped sine
- Apply $x[n]$ to an LTI system with impulse response $h[n]$, Then

$$y[n] = H\{x[n]\} = h[n] * x[n]$$





$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

- If

$$x[n] = z^n$$

we get

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\, z^{n-k}$$

$$y[n] = z^n \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$

- The z-transform is defined as

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$

we may write as

$$H\{z^n\} = H(z)\, z^n$$

You can see that when you do the z-transform it sums up all the sequence, and so the individual terms affect the dependence on *z*, but the resulting function is just a function of *z*, it has no *k* in it. It will become clearer later why we might do this.

- This has the form of an eigen relation, where *zn* is the eigen function and *H(z)* is the eigen value.
- The action of an LTI system is equivalent to multiplication of the input by the complex number 11(z).

- If $1I(z)$ — $111(z)lei^{0}z)$ then the system output is

$$An] — H(z)le-'^{(z)}$$

- Using z — $re\text{-}g^{2}$ we get

$$4n] — 1\ H(rej^{c2})1rn \cos(Q\ n + 1)(re^{-112})+$$

$$Ar^{6112})11r\text{ssiinn}(S^{2}2nn++(11)(reei^{2}$$

- Rewriting $x[n]$

$$x[n] — z' — r'' \cos (\mathbf{S2}\ n) + jr'' \sin(c2n;$$

- If we compare $x[n]$ and $y[n]$, we see that the system modifies
  - the amplitude of the input by $1\ H(re^{-42})\ 1$ and
  - shifts the phase by $4o(reji^{-2})$

DTFT **and the z-transform**

- Put the value of z in the transform then we get

$$H(re^{j\Omega}) = \sum_{n=-\infty} h[n](ref^{2})'$$

$$= \mathbf{I}\ (h[n]r^{n})e^{-g211}$$

- We see that $H(rei^{2})$ corresponds to DTFT of $h[n]r$
- The inverse DTFT of $H(reP)$ must be $h[n]r^{-}n$
- We can write

$$h[n]\,r{-}n — \frac{1}{2rc}\int_{-\tau}^{\pi} II(re^{-112})e\text{-}g^{)1\cdot}\,c/S2$$

The z-transform contd..

- Multiplying $h[n]r^{-}n$ with r' gives

$$h[n] — \frac{}{2m}\int_{-\pi}^{\pi} H(re^{j\Omega})e^{j\Omega n}d\Omega$$

$$h[n] — \frac{1}{2\pi}\int_{-\tau c}^{\pi} H(refi^{-2})(refi^{2})''cic2$$

- We can convert this equation into an integral over z by putting ref° — z
- Integration is over **SI,** we may consider r as a constant

- We have

$$dz = \quad = jzdc2$$

$$= zdz^{1}$$

$$1$$

- uonsider limits on integral
  - — S2 varies from —rc to it
  - z traverses a circle of radius r in a counterclockwise direction
- We can write $h[n]$ as $h[n] = 2Rj\ H(z)zn^{-1}\ dz$ where f is integration around the circle of radius 14 — r in a counter clockwise direction
- The *z-transform* of any signal $x[n]$ is

$$X(Z) - \sum x[n]z^{-n}$$

- The *inverse z-transform* of is

$$x[n] - \frac{1}{27Ej,} X(z)^{-1}\ dz$$

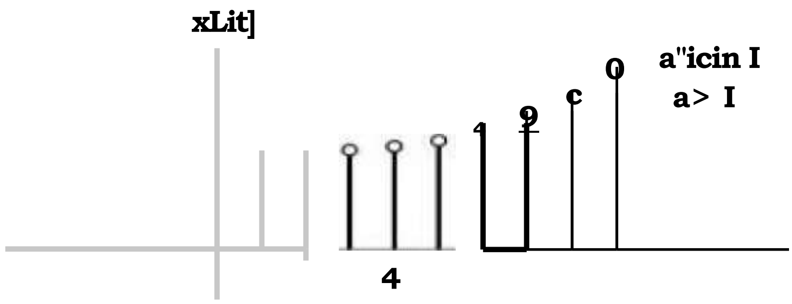
- *Inverse z-transform* expresses $x[n]$ as a weighted superposition of corn plex exponentials
- The weights are $(+_0)X(z)z^{-li}dz$
- This requires the knowledge of complex variable theory

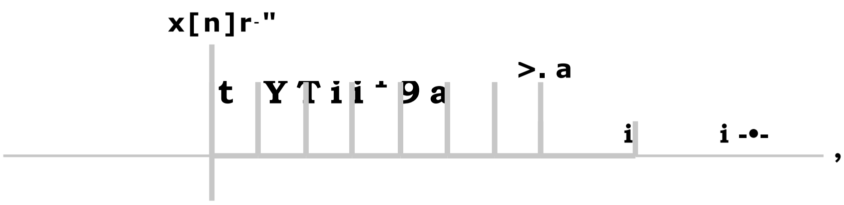
Convergence

- Existence of *z-transform:* exists only if __,*A[n]z-n* converges
- Necessary condition: absolute summability of *x[n] z-n* , since *lx[n]z-n* —
  ~*x[n]r* n1, the condition is

I < '

- The range *r* for which the condition is satisfied is called the *range of convergence* (ROC) of the *z-transform*
- ROC is very important in analyzing the system stability and behavior
- We may get identical *z-transform* for two different signals and only ROC differentiates the two signals
- The *z-transform* exists for signals that do not have DTFT.
- existence of DTFT: absolute summability of *x[n]*
- by limiting restricted values for *r* we can ensure that *x[n]r-n* is absolutely summable even though *x[n]* is not
- Consider an example: the DTFT of *x[n]* — *an u[n]* does not exists for 1α1 > 1
- **If** *r* > a, then r^n decays faster than *x[n]* grows
- Signal *x[n]r-n* is absolutely summable and *z-transform* exists





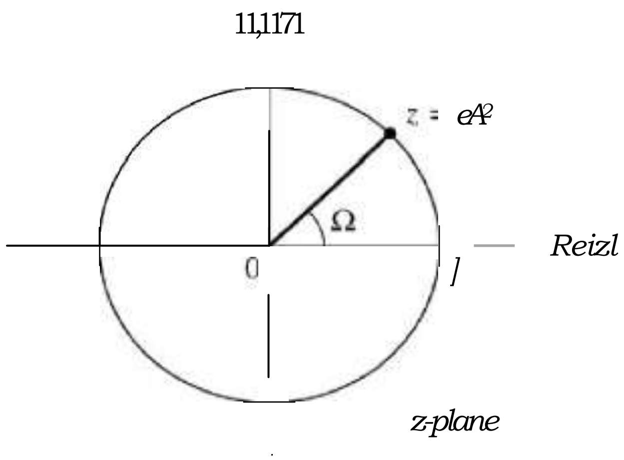


Figure 1.31: DTFT and z-transform

The z-Plane and DTFT

- If *x[n]* is absolutely summable, then DTFT is obtained from the *z-transform* by setting *r— 1(z — e-i),* ie. *X(eP)* — X(z)1,,,12 as shown in Figure ??

Poles and Zeros

- Commonly encountered form of the z-transform is the ratio of two polynomials in z^{-1}

$$X(z) \quad \frac{bo + biz^{\ 1} + \quad + bmz^{-M}}{ao + a1z^{-} + \ \dots \ + bNz^{-N}}$$

- It is useful to rewrite *X(z)* as product of terms involving roots of the numerator and denominator polynomials

$$X(z) \quad \frac{41k/=1(1 \ - clz^{-1})}{fln(1 \ - diz^{-1})}$$

where b — bo/ao

Poles and Zeros contd..

- Zeros: The *ck* are the roots of numerator polynomials
- Poles: The *dk* are the roots of denominator polynomials
- Locations of zeros and poles are denoted by "0" and "x" respectively

Example 1:

- The z-transform and DTFT of $x[n] = \{1,2,-1, 1\}$ starting at $n = -1$

- $X(z) = r,°\_,,,,,,v[n]z n = 1 \quad , \; x[n]z n = \quad z+2 - z^{1} + z^{-z}$

  $X(eP) = X(z)1,,,f2 = ej° + 2 - e \pm e^{-} i^2 Q$

- The z-transform and DTFT of $x[n] = \{1,2,-1,1\}$ starting at $n = -1$

  $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = 1 \; \_ix[n]z^{-n} = z+2 - z^{1} \pm Z^{2}$

- $X(eP) = X(Z)1_2\text{-}02 = eA^2 + 2 - e\text{-})^{12} + eJ^{12}$

Example 2

- Find the z-transform of $x[n]$ — $ocnu[n]$, Depict the ROC and the poles and zeros

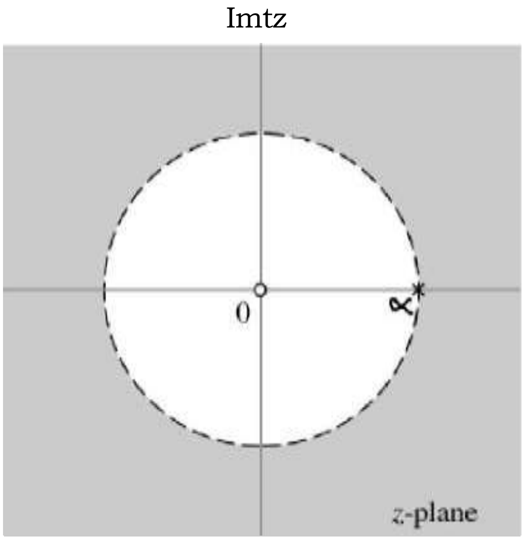
- Solution: $X(z)$ — 17, $otfiu[n]z\text{-}fi$ — E71_0(1)n

  The series converges if Izi > lal

  $X(z)$ — $\frac{1}{1-acre}$ — $\frac{z}{}$ IZI > lal.

  Hence pole at z — oc and a zero at z — 0

■ The ROC is



Properties of Region of Convergence:

- ROC is related to characteristics of *x[n]*
- ROC can be identified from *X(z)* and limited knowledge of x[n]
- The relationship between ROC and characteristics of the *x[n]* is used to find inverse z-transform

Property 1

ROC can not contain any poles

- ROC is the set of all *z* for which z-transform converges
- *X(z)* must be finite for all *z*
- If *p* is a pole, then 1 11(MI — and z-transform does not converge at the pole
- Pole can not lie in the RO(

Property 2

The ROC for a finite duration signal includes entire z-plane except *z* — 0 or/and *z* —

- Let *x[n]* be nonzero on the interval ni < n < *112*. The z-transform is

$$X(z) — \sum_{1x111}^{112} x[n]\ z$$

The ROC for a finite duration signal includes entire z-plane except *z* — 0 or/and *z* —

- If a signal is causal (n2 > 0) then *X(z)* will have a term containing z^{-1}, hence ROC can not include $z = 0$
- If a signal is non-causal (n1 < 0) then *X(z)* will have a term containing powers of z, hence ROC can not include $z =$

The ROC for a finite duration signal includes entire z-plane except z — U or/and $z =$.

- If $n_2 < 0$ then the ROC will include $z — 0$
- If $n_i > 0$ then the ROC will include z —
- This shows the only signal whose ROC is entire z-plane is $x[n] — oS[n]$, where c is a constant

Finite duration signals

- The condition for convergence is IX(z)1 < .

$$IX(z) I — 1\ 1\ x/n/z\ 1$$

$$\sum_{rr\ -\sim} ix[n]z\ 1$$

magnitude of sum of complex numbers < sum of individual magnitudes

fr Magnitude of the product is equal to product of the magnitudes

$$\sum_{n=-\infty} lAinlz\ 111 — \sum_{rr—\sim} 1 \quad n$$

- split the sum into negative and positive time parts
- Let

$$(z) — \sum_{n=—\sim}^{—1} I\ x[n]\ I\ zl\ "$$

$$\pm(z) — \sum_{17=0} I\ x[n]\ I\ zl$$

- Note that $X(z) = I_-(z) + I_+(z)$. If both $I_-(z)$ and $I_+(z)$ are finite, then V($z)$ **if** finite
- If $x[n]$ is bounded for smallest $+ve$ constants A A_+, 1 _ and r_+ such that

$$|x[n]| \le A_-(r_-)^n, \quad n < 0$$

$$|x[n]| \le A_+(r_+)^n \quad n > 0$$

- The signal that satisfies above two bounds grows no faster than $(r_+)^n$ for $+ve$ n and $(r_-)^n$ for $-ve$ n
- If the $n < 0$ bound is satisfied then

$$I_-(z) \le A_- \sum_{n=-\infty}^{-1} (r_-)^n |z|^{-n}$$

$$= A_- \sum \left(\frac{r_-}{|z|}\right)^{n} = A_- \sum_{k=1}^{\infty} \left(\frac{|z|}{r_-}\right)^k$$

- Sum converges if $|z| < r_-$
- If the $n > 0$ bound is satisfied then

$$I_+(z) = A_+ \sum_{n=0} (r_+)^n |z|^{-n}$$

$$= A_+ \sum_{n=0} \left(\frac{r_+}{|z|}\right)^n$$

- Sum converges if $|z| > r_+$
- If $r_+ < |z| < r_-$, then both $I_+(z)$ and $I_-(z)$ converge and $X(z)$ converges

Properties of Z — transform:

— Linearity

— Time reversal

— Time shift

— Multiplication by ce

— Convolution

— Differentiation in the z-domain

The z-transform

- The *z-transform* of any signal *x[n] is*

$$X(z) \; — \; \sum_{n=-\infty} x[n]z^{-n}$$

- The *inverse z-transform* of *X(z)* is

$$K[n] \; — \; \frac{1}{anj,} \;) \; X(z)zn^{-1} \, dz$$

- We assume that

$$x[n] \xrightarrow{z} X(z), \qquad \text{with ROC } R_x$$

$$y[n] \xrightarrow{2} Y(z), \qquad \text{with ROC } R_y$$

- General form of the ROC is a ring in the z-plane, so the effect of an operation on the ROC is described by the a change in the radii of ROC

Pl: Linearity

- The z-transform of a sum of signals is the sum of individual z-transforms

$$min] + byjn] \xrightarrow{Z} > aX(z) +$$

with ROC at least $R_x il \; R_y$

- The ROC is the intersection of the individual ROCs, since the z-transform of the sum is valid only when both converge

P1: Linearity

- The ROC can be larger than the intersection if one or more terms *in* *x[n]* or *y[n]* cancel each other in the sum.

- Consider an example: $x[n] = (1)u[n] \; — (DBu[—n—1]$

- We have $x[n \; < \quad X(z)$

P2: Time reversal

- Time reversal or reflection corresponds to replacing z by z^{-1}-. Hence, if R_A, is of the form $a < z1 < b$ then the ROC of the reflected signal is $a < 1/ 1z1 <$ bor $1/ b < lz < 1/a$

$$\text{If } x[n] \xrightarrow{z} X(z), \quad \text{with ROC } R_„$$

$$\text{Then} \quad x[-12] \xrightarrow{z} x(\tfrac{1}{z}), \quad \text{with ROC } \frac{1}{R_x}$$

Proof: Time reversal

- Let $y[n] — x[-n]$

$Y(z)$ — 17, $x[— n]\ z^{-n}$

Let 1 — $-n$, then

$Y(z)$ — E- xiliz^{i}

$Y(z)$ — ET— x[]] $(1)^{-1}$

$Y(z)$ — $X(1)$

P3: Time shift

- Time shift of n_o in the time domain corresponds to multiplication of z^{-no} in the z-domain

$$\text{If } x[n] \xrightarrow{z} X(z), \quad \text{with ROC } R_x$$

$$\text{Then} \quad x[n - n_o] \xrightarrow{z} z^{-n^{\circ}} X(z),$$

with ROC R,„ except z — 0 or Izi —

P3: Time shift, $n_o > 0$

- Multiplication by z^{-no} introduces a pole of order n_o at z — 0
- The ROC can not include z — 0, even if R_x does include z— 0
- If $X(z)$ has a zero of at least order n_o at z — 0 that cancels all of the new poles then ROC can include z = **0**

P3: Time shift, $n_o < 0$

- Multiplication by z^{-no} introduces n_o poles at infinity
- If these poles are not canceled by zeros at infinity in $X(z)$ then the ROC of $z^{-no}X(z)$ can not include Izi —

Proof: Time shift

- **Let** $y[n] = x[n - n_o]$

  $Y(z) = \_ \ x[n - n_o]$

  Let 1 $=$ - n_o, then

  $Y(z) =$,,,, $x[i]\ z-$ Or-Eno)

  $Y(z) = z-$ no IT _ x-[i] zi

  $Y(z) = z^{-o}X(z)$

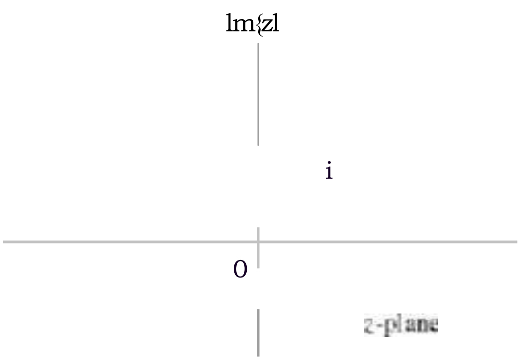
P4: Multiplication by cca

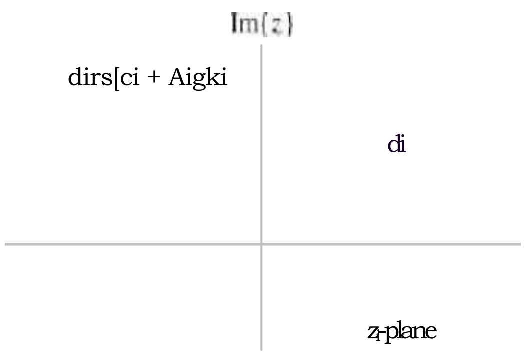
- Let a he a complex number

If $x[n] \xrightarrow{z} X(z)$, with ROC R,

Then $a^n\ x[n] <^z> X(7)$ with ROC lociR$_x$

- loc1R$_x$ indicates that the ROC boundaries are multiplied by locl.
- If R_x is $a < 1z1 < b$ then the new ROC is Iola < 1z1 < Falb
- If $X(z)$ contains a pole d, ie. the factor $(z— d)$ is in the denominator then $X(\zeta)$ has a factor $(z—$ ad) **in** the denominator and thus a pole at ad.
- If $X(z)$ contains a zero c, then X(f,) has a zero at ac
- This indicates that the poles and zeros of $X(z)$ have their radii changed by la
- Their angles are changed by arg{a}





(b)

- If la = 1 then the radius is unchanged and if a is -Eve real number then the angle is unchanged

Proof: Multiplication by a

- Let y $n] = x[n]$

$$Y(z) - I\ cc'\ x[n]z^{-}$$

$$Y(4 - 1^{)}4^{1}R^{-m}_{l=-\infty}$$

$$Y(z) = X(\frac{z}{a})$$

P5: Convolution

- Convolution in time domain corresponds to multiplication in the z-domain If $x[n] \xrightarrow{Z} X(z)$, with ROC R_x If $y[n] <^{Z}> Y(z)$ with ROC R_y
  Then $x[n] * y[n] \xrightarrow{Z} X(z)Y(z)$,
  with ROC at least $R_x \cap R_3$

- Similar to linearity the ROC may be larger than the intersection of R, and R_y

Proof: Convolution

- **Let** $c[n] - x[n] * y[n]$

$$(z) - X\ (x[n] * An])\ z$$

$$C(z) - E\ (\ E_{k}\ x[k] * An - Ic]))\ z$$

$$C(z) - x[k]((\underbrace{E_{rr}\ y[n - Ic])z^{-(n-ft)}}_{Y(z)})z$$

$$C(z) - (\underbrace{y\ X[k]Z^{-k}}_{X(z)})1^{7}(Z)$$

$$C(z) - X(z)Y(z)$$

P6: Differentiation in the z domain

- Multiplication by n in the time domain corresponds to differentiation with respect to z and multiplication of the result by $-z$ in the z-domain

  If $x[n] \xrightarrow{z} X(z)$, with R0(q_x Then $nx[n] \xleftrightarrow{z} -z\,1X(z)$ with ROC R_x

- ROC remains unchanged

Proof: Differentiation in the z domain

- We know

$$X(z) = \sum_{ti=-} x[n]z^{-}$$

Differentiate with respect to z

$$\frac{d}{d}X(z) - I\ (-n)x[n]z^{-n}\ z^{-}$$

- Multiply with $-z$

$$-z\frac{d}{dz}X(z) - \sum_{n=-\infty} - (-n)x[n]z^{-}\ nz^{1}z$$

$$-\frac{d}{T}X(z) - I\ nx[n]z^{-n}$$

Then $nx[n] \longrightarrow -z\frac{d}{i,}X(z)$ with ROC f?,,

Example 1

Use the z-transform properties to determine the z-transform

- $x[n] = n((2^1)au[n])*(1)^{-}uu[-n]$
- Solution is:

$a[n] = (2)^n u[n] \xrightarrow{z} A(z) = \frac{1}{1+1,1,} \quad 1z1 > 2$

$b[n] = na[n] \xleftrightarrow{Z} B(z) = -z\frac{d}{dz}A(z) = -$

$b[n] = na[n] \xleftrightarrow{z} B(z) = \frac{\_1,}{(1+\_{14^1{}_2}z_)^2{}_2{}^{'}{}_{\cdot 1}}$

$c[n] = a_i ru[n] \leftrightarrow C(z) = \frac{}{1}$

Use the z-transform properties to determine the z-transform

- $x[n] = n((^-2^1)bTu[n])*(4^1)^{-}nu[-n]$

$d[n] \quad c[n] = (\frac{1}{4})^{-a} u[n] \longrightarrow D(z) = C(1) = \frac{1}{14^1 z}, \quad Iz/< 4$

$x[n] = (b[n] * d[n]) \longrightarrow X(z) = B(z)D(z), \quad 1 < Izi < 4$

$x[n] = (b[n] * d[n]) \xrightarrow{z} \frac{}{(t+12-)2\ (1-12-)'} \quad <1z1 <4$

$x[n] = (b[n] * d[n]) \xrightarrow{Z} \frac{2z}{(1+\frac{1}{2}z)^2(z-4)}, \quad <Iz/< 4$

Example 2

Use the z-transform properties to determine the z-transform

- $x[n] = a^n \cos(\Omega_o n)u[n]$, where a is real and +ve
- Solution is:

$b[n] = a^n u[n] \longrightarrow B(z) = \frac{}{1-az^{-1}}, \quad Izi>$

Put $\cos(\Omega_o n) = \frac{1}{2}e^{j\Omega_o n} \pm ze^{-1120^{`1}}$, so we get

$x[n] = 26.-Punb[n] \pm e^{-11\text{-}2.'}nb[n]$

Use the z-transform properties to determine the z-transform

- $x[n]$ — ecos(52_0n)u[n], where a is real and -Eve
- Solution continued

$x[n] \xleftrightarrow{z} \quad = \quad B(e^{-4-2}=z) \pm ^1.8(e^{--4-2^o}z), \quad Iz/> a$

$X[n] \xrightarrow{z} X(z) = \frac{i}{2\ 1\ a\ w\ n = 1} + 2\ \frac{1}{1\ o\ z\ \_\ i\ ,\ i\ z\ i >}$

$x[n] \xrightarrow{z} X(z) = 2\left(\left(\frac{1-aej''''z^{-'}+i\text{-}ae\text{-}i^{l2}.}{aeinoz^{-1})(1 -ae^{-}.1^0 0z^{-1})\ 1}\right)\right.$

$x[n] \xleftrightarrow{z} X(z) = \frac{1-a\cos(120)z^{-1}}{1-lac} \quad IZ > a$

Inverse Z transform:

Three different methods are:

1. Partial fraction method
2. Power series method
3. Long division method
4.

Partial fraction method:

- In case of LTI systems, commonly encountered form of z-transform is

$$X(z) = \frac{B(z)}{A(z)}$$

$$X(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + \quad + \dots + a_N z^{-N}}$$

Usually $M < N$

- If $M > N$ then use long division method and express $X(z)$ in the form

$$X(z) \sum_{k=0}^{M-N} f_k z^{-k} + \frac{B(z)}{A(z)}$$

where p(z) now has the order one less than the denominator polynomial and use partial fraction method to find z-transform

- The inverse z-transform of the terms in the summation are obtained from the transform pair and time shift property

$$1 \longleftrightarrow \delta[n]$$

$$z^{-n_o} \longleftrightarrow \delta[n - n_o]$$

- If $X(z)$ is expressed as ratio of polynomials in z instead of z^{-1} then convert into the polynomial of z^{-1}-
- Convert the denominator into product of first-order terms

$$X(z) = \frac{b_0 \pm b_1 z^{-1} + \dots \pm b_M z^{-M}}{a_0 \prod_k (1 - d_k z^{-1})}$$

where d_k are the poles of $X(z)$

For distinct poles

- For all distinct poles, the $X(z)$ can be written as

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{(1 - d_k z^{-1})}$$

- Depending on ROC, the inverse z-transform associated with each term is then determined by using the appropriate transform pair
- We get

$$A_k (d_k)^n u[n] \longleftrightarrow \frac{A_k}{1 - d_k z^{-1}}$$

with ROC $z > dk$ OR

$$-Ak(dk)^n u[-n-1] \quad \frac{z \quad Ak}{1 - clkz^{-1}}$$

with ROC $z < d_k$

- For each term the relationship between the ROC associated with $X(z)$ and each pole determines whether the right-sided or left sided inverse transform is selected

For Repeated poles

- **If** pole di is repeated r times, then there are terms in the partial-fraction expansion associated with that pole

$$\frac{Ai,}{1 - dz\,t'} \quad \frac{A_{i},-}{(1 - \quad t)^{2}}' \quad \frac{}{(1 - diZ^{-1})^{r}}$$

- Here also, the ROC of $X(z)$ determines whether the right or left sided inverse transform is chosen.

$$A \frac{\quad - 1}{(m-1)!}(di)n \qquad \frac{z}{(1 - diz^{-1})^{171.}} \qquad \text{with ROC } 1z1 >$$

- **If** the ROC is of the form Izi $d_{,}$, the left-sided inverse z-transform is chosen, ie.

$$-A\,\frac{(n+1) \quad \dots \quad (n \pm m \quad - \quad -1)(di)^n u[-I1 - 1]}{(m - 1)!} \qquad \frac{A}{(1 - dz^{-1})^{177}} \qquad \text{with ROC} 1z1 <$$

Deciding ROC

- The ROC of $X(z)$ is the intersection of the ROCs associated with the individual terms in the partial fraction expansion.
- In order to chose the correct inverse z-transform. we must infer the ROC of each term from the ROC of $X(z)$.
- By comparing the location of each pole with the ROC of $X(z)$.
- Chose the right sided inverse transform: if the ROC of $X(z)$ has the radius greater than that of the pole associated with the given term
- Chose the left sided inverse transform: if the ROC of $X(z)$ has the radius less than that of the pole associated with the given term

Partial fraction method

- It can be applied to complex valued poles
- Generally the expansion coefficients are complex valued
- If the coefficients in $X(z)$ are real valued, then the expansion coefficients corresponding to complex conjugate poles will he complex conjugate of each other

- Here we use information other than ROC to get unique inverse transform
- We can use causality, stability and existence of DTFT
- If the signal is known to be causal then right sided inverse transform is chosen
  - If the signal is stable, then t is absolutely summable and has DTFT
  - Stability is equivalent to existence of DTFT, the ROC includes the unit circle in the z-plane, ie. IA = 1
  - The inverse z-transform is determined by comparing the poles and the unit circle
  - If the pole is inside the unit circle then the right-sided inverse z-transform is chosen
  - If the pole is outside the unit circle then the left-sided inverse z-transform is chosen

Power series expansion method

- Express *X(z)* as a power series in z^{-1} or *z* as given in z-transform equation
- The values of the signal *x[n]* are then given by coefficient associated with z^{-n}
- Main disadvantage: limited to one sided signals
- Signals with ROCs of the form 121 > *a* or 1z1 < a
- If the ROC is IA > *a,* then express *X(z)* as a power series in z we get $^{-1}$ and right sided signal
- If the ROC is 1z1 < *a,* then express *X(z)* as a power series in *z* and we get left sided signal

Long division method:

- Find the z-transform of

$$X(z) = \frac{2 + z^{-1}}{1 - P—1,} \text{ with ROC } 1z1 >$$

- Solution is: use long division method to write $X(z)$ as a power series in z^{-1}, since ROC indicates that $x[n]$ is right sided sequence

- We get

$$X(z) = 2 + 2zz^{-1} \pm \pm —z-- + .$$
2

- Compare with z-transform

$$X(z) = \text{I} \; x[n]$$

- We get

$$x[n] = 26[n] + 28[n-1] + 5[n-2]$$

$$+—_2[n-3]+$$

- If we change the ROC to < 1, then expand $X(z)$ as a power series in z using long division method
- We get

$$X(z) — -2 - \underline{8z} - 16z^2 - 32z^3 +$$

- We can write $x[n]$ as

$$r[n] — -26[n] - 86[n+1] - 166[n+2]$$

$$- 326[n \pm 3] \pm$$

- Find the z-transforrn of

$$X(z) = e^{2}, \text{ with ROC all } z \text{ except } 121 = —$$

- Solution is: use power series expansion for ea and is given by

$$e^{a} = \sum_{k=0} \frac{a}{}$$

- We can write $X(z)$ as

$$X(z) = \sum_{k=0}^{\infty} \frac{(z^2)^k}{}$$

$$X(z\} = \mathrm{E}_{k=0} \frac{}{ki}$$

- We can write $x[n]$ as

$$r[n] — \begin{cases} 0 & n > 0 \text{ or } n \text{ is odd} \\ \frac{1}{()}^{1}' & \text{otherwise} \end{cases}$$

Recommended Questions

1. Using appropriate propertes fmd the Z-transform of $x(n)=n^2(1/3)^n u(n-2)$

2. Determine the inverse Z- transform of $X(z)=1/(2-z^{-1} +2 z^{-2)}$ by long division method

3. Determine all possible signals of x(n) associated with Z- transform
$X(z)= (1/4) z^{-1} / [1-(1/2) z^{-1}][1-(1/4) z^{-1}]$

4. State and prove time reversal property. Find value theorem of Z-transform. Using suitable properties, fmd the Z-transform of the sequences
   i) (n-2)(1/3)" u(n-2)
   ii) $(n+1)(1/2)^{n\pm1} \cos w_o(n+1) u(n+1)$

5. Consider a system whose difference equation is y(n - 1) + 2y(n) = x(n)
   i) Determine the zero-input response of this system, if y(-1) = 2.
   ii) Determine the zero state response of the system to the input x(n)=(114t u(n).
   iii) What is the frequency response of this system?
   iv) Find the unit impulse response of this system.

NIT 8: Z-Transforms — 2 **Teaching hours: 6**

Z-transforms — 2: Transform analysis of LTI Systems, unilateral Z Transform and its application to solve difference equations.

TEXT BOOK

Simon Haykin and Barry Van Veen "Signals and Systems", John Wiley & Sons, 2001.Reprint 2002

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UNIT 8
Z-Transforms – 2

8.1 Transform analysis of LTI systems:

- We have defined the transfer function as the z-transform of the impulse response of an LTI system

$$H(z) = \sum_{k=-\infty} h[k]z^{-k}$$

- Then we have $y[n] = x[n] * h[n]$ and $Y(z) = X(z)H(z)$
- This is another method of representing the system
- The transfer function can be written as

$$H(z) = \frac{Y(z)}{X(z)}$$

- This is true for **all** z in the ROCs of $X(z)$ and $Y(z)$ for which $X(z)$ in nonzero
- The impulse response is the z-transform of the transfer function
- We need to know ROC in order to uniquely find the impulse response
- If ROC is unknown, then we must know other characteristics such as stability or causality in order to uniquely find the impulse response

System identification

- Finding a system description by using input and output is known as system identification
- Exl : find the system, if the input is $x[n] = (-1/3)^n u[n]$ and the out is $y[n] = 3(-1)^n u[n] + (1/3)^n u[n]$

- Solution: Find the z-transform of input and output. Use *X(z)* and Y(z) to find *H(z)*, then find *h(n)* using the inverse z-transform

$$X(z) \quad \frac{1}{(1 \pm (D z^{-1})'} \qquad \text{with ROC } 1z1 > \; -$$

$$Y(z) \quad \frac{3}{(1 - rz^{-1})} \pm \frac{1}{(1 - (A)z^{-1})'} \qquad \text{with R0(} \quad Izi > 1$$

- We can write Y(z) as

$$Y(z) \quad \frac{4}{(1 + z^{-1})(1 - (3 \;)z^{1})} \qquad \text{with ROC } 1z1 > 1$$

- We know $H(z) \; - \; Y(z) / X(z)$, so we get

$$1/(z) \quad \frac{4(1 + (4)z^{-1})}{(1 \pm z^{-1})(1 - (Dz^{-1})} \qquad \text{with ROC } 1z1 \qquad 1$$

- We need to find inverse z-transform to find *x[n]*, so use partial fraction and write *H(z)* as

$$H(z) = \frac{2}{1 \pm z^{-1}} + \frac{2}{1 - (A) z^{1}} \qquad \text{with ROC } 1z1 > 1$$

- Impulse response *x[n]* is given by

$$h[n] = \; -1)^{n} u[n] \pm 2(113)^{n} u[n]$$

Relation between transfer function and difference equation

- The transfer can be obtained directly from the difference-equation description of an LTI system
- We know that

$$\sum_{k=0}^{N} a_k y[n - k] = \sum_{k=0}^{M} b_k x[n - k]$$

- We know that the transfer function $11(z)$ is an eigen value of the system associated with the eigen function zn, ie. if 4n] = z" then the output of an LTI system $y[n] = z^{n} H(z)$
- Put $x[n - k] = {}^{k}$ and $y\; n \; - k] = z^{n} 11(z)$ in the difference equation,

we get

$$z\sum_{k=0}^{N} a_k z^{-k} H(z) = z\sum_{k=0}^{M} b_k z^{-k}$$

- We can solve for $H(z)$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- The transfer function described by a difference equation is a ratio of polynomials in z^{-1}- and is termed as a rational transfer function.
- The coefficient of z^{-k} in the numerator polynomial is the coefficient associated with $x[n-k]$ in the difference equation
- The coefficient of z^{-k} in the denominator polynomial is the coefficient associated with $y[n-k]$ in the difference equation
- This relation allows us to find the transfer function and also find the difference equation description for a system, given a rational function

Transfer function:

- The poles and zeros of a rational function offer much insight into LTI system characteristics
- The transfer function can be expressed in pole-zero form by factoring the numerator and denominator polynomial
- If c_k and d_k are zeros and poles of the system respectively and $b = b_0/a_0$ is the gain factor, then

$$H(z) = \frac{b\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

- This form assumes there are no poles and zeros at $z = 0$
- The p^t order pole at $z = 0$ occurs when $b_0 = b_1 = \ldots = b_{p-1} = 0$
- The order zero at $z = 0$ occurs when $a_0 = a_1 = \ldots = a_{l-1} = 0$
- Then we can write $H(z)$ as

$$H(z) = \frac{b z^{-p}\prod_{k=1}^{M-p}(1 - c_k z^{-1})}{z^{-l}\prod_{k=1}^{N-l}(1 - d_k z^{-1})}$$

where $b = b_p/a_l$

- In the example we had first order pole at $z = n$
- The poles, zeros and gain factor b uniquely determine the transfer function
- This is another description for input-output behavior of the system
- The poles are the roots of characteristic equation

8.2 Unilateral Z- transforms:

- Useful in case of causal signals and LTI systems
- The choice of time origin is arbitrary, so we may choose n — U as the time at which the input is applied and then study the response for times $n > \mathbf{0}$

Advantages

- We do not need to use ROCs
- It allows the study of LTI systems described by the difference equation with initial conditions

Unilateral *z*-transform

- The unilateral z-transform of a signal $x[n]$ is defined as

$$X(z) — y \; x[n]z \quad n=0$$

which depends only on $x[n]$ for $n > 0$

- The unilateral and bilateral z-transforms are equivalent for causal signals

$$\text{ocnum]} \; \frac{z}{\quad} \; \frac{1}{1 — ocz^{-1}}$$

$$a' \cos(i2_{„}n)u[n] \; z_u \; \frac{1 — a\cos(c2_0)z^{-1}}{1 — 2a\cos(fl_o)z^{-1} + a^2 z^{-2}}$$

Properties of unilateral Z transform:

- Consider the difference equation description of an LTI system

$$\sum_{k=0}^{N} a_k y[n - k] — \sum_{k=0}^{m} b_k x[n - k]$$

- We may write the z-transform as

$$A(z)Y(z) + C(z) — B(z)X(z)$$

where

$$A(z) — \sum_{k=0} a_k z^{-k} \quad \text{and} \quad B(z) — \sum_{k=0}^{M} b_k z^{-k}$$

- The same properties are satisfied by both unilateral and bilateral z-transforms with one exception: the time shift property

- The time shift property for unilateral z-transform: Let $w[n] - x[n - 1]$
  - The unilateral z-transform of w[n] is

$$W(z) — I_{11-(} w[n] \ z '' — E_{n=o} \ x[n - 1] z—$$

$$W(z) — x1 — \ _{II=1} ... - 1]2=r_i$$

$$W(Z) = X — 11 \pm y \ x[m]z^{-(-1-1)}_{m==0}$$

- The unilateral z-transform of $w[n]$ is

$$W(z) — \ - 1] + z^{1} \ X_{m—o} \ x[m]$$

$$W(z) — x[- 1] \pm z^{1} - X(z)$$

- A one-unit time shift results in multiplication by z^{-1} and addition of the constant $x[-1]$
- In a similar way, the time-shift property for delays greater than unity is

$$x[n - k] \xrightarrow{u} x[— \quad x[— k + 1] z^{-1} +$$

$$+x[- 1]z^{-k+1} \pm Z^{-k}X(4 \text{ for } k > 0$$

- In the case of time advance, the time-shift property changes to

$$x[n + lc] \xrightarrow{u} -x[0]z^{k} - 4 - 1izk^{-1} \pm$$

$$\ldots - \quad - l]z + z^{k}X(z) \text{ for } k > 0$$

8.3 Application to solve difference equations

Solving Differential equations using initial conditions:

- We get

$$C(z) - \sum_{n7=0}^{N-1} \sum_{k=n7\pm1} a_k y[-k \pm in] z^{-m}$$

- We have assumed that $x[n]$ is causal and

$$x[n-k] \longrightarrow {}^{k} X(z)$$

- The term $C(z)$ depends on the N initial conditions y[—1] 0[-2] —— y_—N] and the a_k
- $C(z)$ is zero if all the initial conditions are zero

- Solving for $Y(z)$, gives

$$Y(z) = \frac{B(z)}{4(z)} X(z) - \frac{C(z)}{A(z)}$$

- The output is the sum of the forced response due to the input and the natural response induced by the initial conditions
- The forced response due to the input

$$\frac{B(z)}{A(A} X(z)$$

- The natural response induced by the initial conditions

$$\frac{0(z)}{A(z)}$$

- $C(z)$ is the polynomial, the poles of the natural response are the roots of $A(z)$, which are also the poles of the transfer function

- The form of natural response depends only on the poles of the system, which are the roots of the characteristic equation

First order recursive system

- Consider the first order system described by a difference equation

$$y[n] - py[n-1] - x[n]$$

where p $- 1 \pm r/100$, and r is the interest rate per period in percent and $y[n]$ is the balance after the deposit or withdrawal of $x[n]$

- Assume bank account has an initial balance of $10,000!- and earns 6% interest compounded monthly. Starting in the first month of the second year, the owner withdraws $100 per month from the account at the beginning of each month. Determine the balance at the start of each month

- Solution: Take unilateral z-transform and use time-shift property *Am* get

$$Y(z) - p(y[-1] \pm z^{-1} 1'(z)) = X(z)$$

- Rearrange the terms to find Y(z), we get

$$(1 - pz^{1})Y(z) = X(z) \pm p\, y[-1]$$

$$Y(z) = \frac{X(z)}{1 - pz^{-1}} \pm \frac{P[-1]}{1 - pz^{1}}$$

Y(z) consists of two terms

- one that depends on the input: the forced response of the system
- another that depends on the initial conditions: the natural response Df the system

- The initial balance of $10,000 at the start of the first month is the initial condition A-1 |, and there is an offset of two between the time index *n* and the month index
- *y n]* represents the balance in the account at the start of the n + 2^{nd} month.
- We have p = $1 + \frac{}{100} = 1.005$
- Since the owner withdraws $100 per month at the start of month 13 *(11 = 11)*
- We may express the input to the system as x[n] = — 100 *u[n* — 11], we get

$$X(z) \frac{-100z^{-11}}{1-z^{-1}}$$

- We get

$$Y(z) \frac{-100z^{-11}}{(1-z^{-1})(1-1.005z^{-1})} + \frac{1.005(10,000)}{1-1.005z}$$

- After a partial fraction expansion we get

$$Y(z) \frac{20,000z^{-11}}{1-z^{-1}} + \frac{20,000z^{-11}}{1-1.005z^{-1}} + \frac{10,050}{1-1.005z^{-1}}$$

- Monthly account balance is obtained by inverse z-transforming *Y(z)*
  We get

$$y[n] \; -20,000u[n-11] \; -20,000(1.005)^{'11}u[n-11]$$

$$\pm 10,050(1\_005)^n u[n]$$

- The last term 10, 050(1.005)Thu[n] is the natural response with the initial balance
- The account balance
- The natural balance
- The forced response

Recommended Questions

1. Find the inverse Z transform of

$$H(z)= \frac{1+z^{-1}}{(1 - 0.9e^{j\pi/4}z^{-1})(1- 0.9e^{-j\pi/4}z^{-1})}$$

2. A system is described by the difference equation

$$Y(n) - y(n - 1) + \frac{1}{4} y(n - 2) = x(n) + 1/4\, x(n - 1) - 1/8\, x(n - 2)$$

Find the Transfer function of the Inverse system
Does a stable and causal Inverse system exists

3. Sketch the magnitude response for the system having transfer functions.

4. Find the z-transform of the following x[n]:

(a) x[n] - (1,1,- }
(b) x[n] 26(n + 2] - 38(n 2]
(c) x[n] 3(_ 1 2)' [n1- (3Y -n - 1]
(d) x[n] - 2(DI tit -n - 1]

5. Given

$$(z) \quad \frac{z(z - 4)}{(z \quad 2)(z \quad 3)}$$

(a) State all the possible regions of convergence.
(b) For which ROC is X (z) the z-transform of a causal sequence?

6. Show the following properties for the z-transform.

(a) If x[n] is even, then X(z - X(2),
(b) If x(n) is odd, then $X(z^{-1})$- -X(z),
If x[n] odd, then there is a zero in X(z) at z 1.

7. Derive the following transform

$$(\cos \Omega_o n)u[n] \longleftrightarrow \frac{—(\cos \Omega_0) z}{z^2 \quad (2\cos \Omega_0)2 + 1}$$

$$(\sin \Omega_o n)u[n] \longleftrightarrow \frac{(\sin \Omega_0)z}{—}$$

8. Find the z-transforms of the following x[n]:

(a) x[n] = (n - 3)utn -
(b) x[n] (n -
(c) x[n] = - - 3]
(d) x[n] = u[n - 3])

9. Using the relation

an4 n ı ⟷ $\frac{z}{z-a}$ 1:1>1u

fmd the z-transform of the following *x[n]:*

(a) *na' 1 u[n]*

(b) An] *n(n - 1)a·"ᶻu[n]*

x[n] *n*(n 1) • . • *(n* - -- I *41"-' Li[]*n

10. Using the z-transform

(a) *x(n]* an]* - *x[n]*

(b) *x(rd* 5(n* - n_0] = $x[n - n_0]$

H. Find the inverse z-transform of X(z)= e^{ak} , z > 0

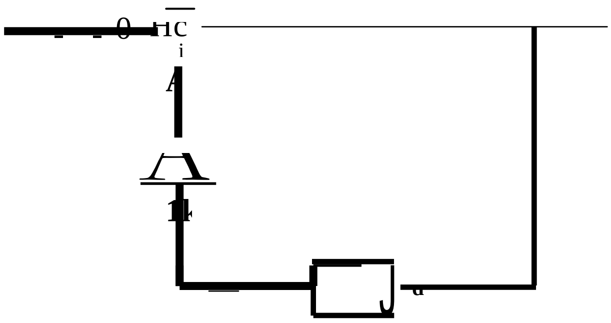
12. Using the method of long division, fmd the inverse z-transform of the following *X (z):*

(a) X.(z) $\frac{z}{(z - \quad z - 2)}$, 121 < 1

(b) X(z)- $\frac{}{(z \quad 1)(z - 2)}$, 1 <1z1<

X(z) $\frac{z}{(z - 1)(: - 2)}$ı 1z1> 2

13. Consider the system shown in Fig. 4-9. Find the system function ***H (z)*** and its impulse response h[n]



14. Consider a discrete-time LTI system whose system function ***H (z)*** is given **by**

H(z) = ——— **d '**)

(a) Find the step response s[n].
(b) Find the output y[n] to the input x[n] = nu[n].

15. Consider a causal discrete-time system whose output y[n] and input x[n] are related by

' - - ÷ ***- 21 -*** '

(a) Find its system function I-1(z).
(b) Find its impulse response h[n].