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- GTU Syllabus

GTU Papers

GUJARAT TECHNOLOGICAL UNIVERSITY

AUTOMOBILE ENGINEERING (02), INDUSTRIAL ENGINEERING (15) & MECHANICAL ENGINEERING (19)

COMPLEX VARIABLES AND NUMERICAL METHODS
SUBJECT CODE: 2141905
B.E. 4th SEMESTER

Type of course: Engineering Mathematics

Prerequisite: As a pre-requisite to this course students are required to have a reasonable mastery over multivariable calculus, differential equations and Linear algebra

Rationale:
Mathematics is a language of Science and Engineering.

Teaching and Examination Scheme:

Teaching Scheme			Credits	Examination Marks						Total Marks
L	T	P	C	Theory Marks			Practical Marks			
				ESE (E)	PA (M)		PA (V)		PA (I)	
					PA	ALA	ESE	OEP		
3	2	0	5	70	20	10	30	0	20	150

Content:

Sr. No.	**Content**	**Total Hrs**	**% Weightage**
1	Complex Numbers and Functions: Exponential, Trigonometric, De Moivre’s Theorem, Roots of a complex number ,Hyperbolic functions and their properties, Multi-valued function and its branches: Logarithmic function and Complex Exponent function Limit ,Continuity and Differentiability of complex function, Analytic functions, Cauchy-Riemann Equations, Necessary and Sufficient condition for analyticity, Properties of Analytic functions, Laplace Equation, Harmonic Functions, Harmonic Conjugate functions and their Engineering Applications	10	24
2	Complex Integration: Curves, Line Integral(contour integral) and its properties, Cauchy-Goursat Theorem, Cauchy Integral Formula, Liouville Theorem (without proof), Maximum Modulus Theorems(without proof)	04	10
3	Power Series: Convergence(Ordinary, Uniform, Absolute) of power series, Taylor and Laurent Theorems (without proof), Laurent series expansions, zeros of analytic functions , Singularities of analytic functions and their classification Residues: Residue Theorem, Rouche’s Theorem (without proof)	05	12
4	Applications of Contour Integration: Evaluation of various types of definite real integrals using contour	02	5

	integration method		
5	Conformal Mapping and its Applications: Conformal and Isogonal mappings , Translation, Rotation & Magnification, Inversion, Mobius(Bilinear) , Schwarz-Christoffel transformations	03	7
6	Interpolation: Finite Differences, Forward, Backward and Central operators, Interpolation by polynomials: Newton's forward ,Backward interpolation formulae, Newton's divided Gauss & Stirling's central difference formulae and Lagrange's interpolation formulae for unequal intervals	04	10
7	Numerical Integration: Newton-Cotes formula, Trapezoidal and Simpson's formulae, error formulae, Gaussian quadrature formulae	03	7
8	Solution of a System of Linear Equations: Gauss elimination, partial pivoting , Gauss-Jacobi method and Gauss-Seidel method	03	7
9	Roots of Algebraic and Transcendental Equations : Bisection, false position, Secant and Newton-Raphson methods, Rate of convergence	03	7
10	Eigen values by Power and Jacobi methods	02	4
11	Numerical solution of Ordinary Differential Equations: Euler and Runge-Kutta methods	03	7

Suggested Specification table with Marks (Theory):

Distribution of Theory Marks				
R Level	U Level	A Level	N Level	E Level
10%	15%	20%	20%	35%

Legends: R: Remembrance; U: Understanding; A: Application, N: Analyze and E: Evaluate and above Levels (Revised Bloom's Taxonomy)

Note: This specification table shall be treated as a general guideline for students and teachers. The actual distribution of marks in the question paper may vary slightly from above table

Reference Books:

1. R. V. Churchill and J. W. Brown, Complex Variables and Applications (7th Edition), McGraw-Hill (2003)
2. J. M. Howie, Complex Analysis, Springer-Verlag(2004)
3. M. J. Ablowitz and A.S. Fokas, Complex Variables-Introduction and Applications, Cambridge University Press, 1998 (Indian Edition)
4. E. Kreyszig, Advanced Engineering Mathematics(8th Edition), John Wiley (1999)
5. S. D. Conte and Carl de Boor, Elementary Numerical Analysis-An Algorithmic Approach (3rd Edition), McGraw-Hill, 1980
6. C.E. Froberg, Introduction to Numerical Analysis (2nd Edition), Addison-Wesley,1981
7. Gerald C. F. and Wheatley,P.O., Applied Numerical Analysis (Fifth Edition), Addison-Wesley, Singapore, 1998.
8. Chapra S.C, Canale, R P, Numerical Methods for Engineers **,** Tata McGraw Hill, 2003

Course Outcome:

After learning the course the students should be able to:

- evaluate exponential, trigonometric and hyperbolic functions of a complex number
- define continuity, differentiability, analyticity of a function using limits. Determine where a function is continuous/discontinuous, differentiable/non-differentiable, analytic/not analytic or entire/not entire.
- determine whether a real-valued function is harmonic or not. Find the harmonic conjugate of a harmonic function.
- understand the properties of Analytic function.
- evaluate a contour integral with an integrand which have singularities lying inside or outside the simple closed contour.
- recognize and apply the Cauchy's integral formula and the generalized Cauchy's integral formula.
- classify zeros and singularities of an analytic function.
- find the Laurent series of a rational function.
- write a trigonometric integral over $[0, 2\pi]$ as a contour integral and evaluate using the residue theorem.
- distinguish between conformal and non conformal mappings.
- find fixed and critical point of Bilinear Transformation.
- calculate Finite Differences of tabulated data.
- find an approximate solution of algebraic equations using appropriate method.
- find an eigen value using appropriate iterative method.
- find an approximate solution of Ordinary Differential Equations using appropriate iterative method.

List of Open Source Software/learning website:

http://ocw.mit.edu/resources/res-18-008-calculus-revisited-complex-variables-differential-equations-and-linear-algebra-fall-2011/part-i/
http://nptel.ac.in/courses/111105038/
http://nptel.ac.in/courses/111104030/
http://nptel.ac.in/courses/111107063/
http://nptel.ac.in/courses/111101003/

ACTIVE LEARNING ASSIGNMENTS: Preparation of power-point slides, which include videos, animations, pictures, graphics for better understanding theory and practical work – The faculty will allocate chapters/ parts of chapters to groups of students so that the entire syllabus to be covered. The power-point slides should be put up on the web-site of the College/ Institute, along with the names of the students of the group, the name of the faculty, Department and College on the first slide. The best three works should submit to GTU.

Complex Number

A number $z = x + iy$ is called a complex number, where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$.

- ✓ x is called the real part of z and is denoted by $Re(z)$.
- ✓ y is called the imaginary part of z and is denoted by $Im(z)$.

Conjugate of a Complex Number

Conjugate of a complex number $z = x + iy$ is denoted by $\bar{z}$ and is defined by $\bar{z} = x - iy$.

Two complex number $x + iy$ and $x - iy$ are said to be complex conjugate of each other.

Arithmetic Operations Of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers then

Addition

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

Subtraction

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$$

Multiplication

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

Division

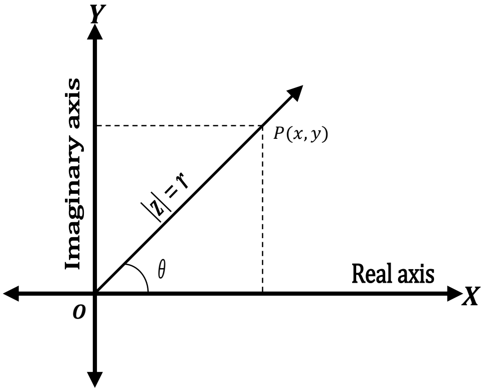
$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

Properties

Let z_1 and z_2 be two complex numbers then

- ✓ $\overline{(\overline{z_1})} = z_1$
- ✓ $|z_1| = |\overline{z_1}|$
- ✓ $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$
- ✓ $\overline{z_1 . z_2} = \overline{z_1} . \overline{z_2}$
- ✓ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}; z_2 \neq 0$
- ✓ $\frac{z_1 + \overline{z_1}}{2} = Re(z_1)$
- ✓ $\frac{z_1 - \overline{z_1}}{2i} = Im(z_1)$
- ✓ $z_1 \cdot \overline{z_1} = x^2 + y^2 = |z|^2$

Geometrical Representation Of Complex Number



Let XOY be a complex plane which is also known as Argand Plane, where $\overleftrightarrow{OX}$ and $\overleftrightarrow{OY}$ are called Real axis and Imaginary axis respectively.

The ordered pair $P(x,y)$ represents the complex number $z = x + iy$.

$\overline{OP}$ represents the distance between complex numbers P and O, it is called **modulus** of z and denoted by $|z|$.

$$\mathbf{i.e.}\ |\mathbf{z}| = \mathbf{r} = \sqrt{\mathbf{x^2 + y^2}} = \sqrt{\mathbf{z.\bar{z}}}$$

Let $\overline{OP}$ makes an angle θ with positive real axis, it is called argument of z.

$$\mathrm{i.e.}\ \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Rules To Determine Argument of a non-zero Complex Number

- ✓ If $x > 0 \ \& \ y > 0\ ,\theta = tan^{-1}\left(\frac{y}{x}\right)$
- ✓ If $x > 0 \ \& \ y < 0\ ,\theta = -\,tan^{-1}\left|\frac{y}{x}\right|$
- ✓ If $x < 0 \ \& \ y > 0\ ,\theta = \pi - tan^{-1}\left|\frac{y}{x}\right|$
- ✓ If $x < 0 \ \& \ y < 0\ ,\theta = -\pi + tan^{-1}\left|\frac{y}{x}\right|$

Notes

- ✓ If $-\pi < \theta \leq \pi$, then argument of z is called “PRINCIPAL ARGUMENT” of z. It is denoted by $Arg(z)$.

$$\boldsymbol{i.e.\ Arg(z) = tan^{-1}\left(\frac{y}{x}\right)}$$

- ✓ $Arg(z)$ is a Single-Valued Function.
- ✓ The “GENERAL ARGUMENT” of argument of z is denoted by “$arg(z)$”.
- ✓ Relation between “$arg(z)$” and “$Arg(z)$”.

$$\boldsymbol{arg(z) = Arg(z) + 2k\pi\ ;\ k = 0 \pm 1, \pm 2, \ldots}$$

- ✓ $arg(z)$ is a Multi-Valued Function.
- ✓ For $z = 0 = 0 + i0$, argument is not defined.

Absolute value or Modulus of a complex number

If $z = x + iy$ is a given complex number then absolute value of modulus of z is denoted by $|z|$ and is defined by $\sqrt{x^2 + y^2}$

$$\mathbf{i.e.}\ |\boldsymbol{z}| = \sqrt{\boldsymbol{x^2 + y^2}} = \sqrt{\boldsymbol{z \cdot \bar{z}}}$$

Properties

- ✓ $|z_1 + z_2| \leq |z_1| + |z_2|$
- ✓ $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

- ✓ $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- ✓ $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- ✓ $z \cdot \bar{z} = |z|^2$

Polar Representation of a Complex Number

Let $z = x + iy$ be a complex number.

Let $x = \cos\theta$ and $y = \sin\theta\,;\ \theta \in (-\pi, \pi]$.

Now, $z = x + iy$

$$= r\cos\theta + i\,r\sin\theta = r(\cos\theta + i\sin\theta)$$

Thus, $z = r(\cos\theta + i\sin\theta)$ is called Polar representation of a complex number.

Where, $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Now, $z = u(r, \theta) + i\,v(r, \theta) = r(\cos\theta + i\sin\theta)$

$\Rightarrow z = u(r, \theta) + i\,v(r, \theta) = r\cos\theta + i\,r\sin\theta$

Thus, $\mathrm{Re}(z) = u(r, \theta) = r\cos\theta$ & $\mathrm{Im}(z) = v(r, \theta) = r\sin\theta$

Exponential Representation Of a Complex Number

By Polar representation,

$$z = r(\cos\theta + i\sin\theta)$$

By Euler Formula, $e^{i\theta} = \cos\theta + i\sin\theta$

Then, $z = re^{i\theta}$ is called Exponential representation.

De-Moivre's Theorem

Statement: $(\cos\theta + i\sin\theta)^{\,n} = \cos n\theta + i\sin n\theta\ ; n \in \mathbb{Q}\left[\text{i.e. } \left(e^{i\theta}\right)^n = e^{in\theta}\right]$

Remarks

- ✓ $(\cos\theta - i\sin\theta)^{\,n} = \cos n\theta - i\sin n\theta$
- ✓ $(\sin\theta \pm i\cos\theta)^{\,n} \neq \sin n\theta \pm i\cos n\theta$
- ✓ $(\cos\theta \pm i\sin\alpha)^{\,n} \neq \cos n\theta \pm i\sin n\alpha$
- ✓ $(\sin\theta \pm i\cos\theta)^{\,n} = \left[\cos\left(\frac{\pi}{2} - \theta\right) \pm i\sin\left(\frac{\pi}{2} - \theta\right)\right]^n = \cos n\left(\frac{\pi}{2} - \theta\right) \pm i\sin n\left(\frac{\pi}{2} - \theta\right)$

Exercise-1

C	Que.1	Find the Real & Imaginary part of $f(z) = z^2 + 3z$. $\left[\mathbf{Re}\left(\mathbf{f(z)}\right) = \mathbf{x^2 + 3x - y^2}, \mathbf{Im}\left(\mathbf{f(z)}\right) = \mathbf{2xy + 3y}\right]$	Jun-13
H	Que.2	Write function $f(z) = z + \frac{1}{z}$ in $f(z) = u(r, \theta) + iv(r, \theta)$ form. $\left[\mathbf{u(r, \theta)} = \left(\mathbf{r} + \frac{\mathbf{1}}{\mathbf{r}}\right)\mathbf{\cos\theta}\ \&\ \mathbf{v(r, \theta)} = \left(\mathbf{r} - \frac{\mathbf{1}}{\mathbf{r}}\right)\mathbf{\sin\theta}\right]$	Dec-13

C	Que.3	Find the value of $\operatorname{Re}\big(f(z)\big)$ and $\operatorname{Im}\big(f(z)\big)$ at the indicated point where $f(z)=\frac{1}{1-z}$ at $7+2i$. $\left[\mathbf{Re\big(f(z)\big)-\frac{3}{20}\ ;\ Im\big(f(z)\big)=\frac{1}{20}}\right]$	Jun-10
T	Que.4	Write function $f(z)=2iz+6\bar{z}$ in $f(z)=u(r,\theta)+iv(r,\theta)$ form. $[\mathbf{u(r,\theta)=6r\cos\theta-2r\sin\theta\ \ \&\ \ v(r,\theta)=2r\sin\theta-6r\cos\theta}]$	Jun-14
H	Que.5	Separate real and imaginary parts of $\sinh z$. $[\mathbf{Re(\sinh z)=\sinh x\cos y\ ;\ Im(\sinh z)=\cosh x\sin y}]$	Jun-14
C	Que.6	Find real and imaginary part of $(-1-i)^7+(-1+i)^7$. $[\mathbf{Re(z)=-16, Im(z)=0}]$	Jun-11
C	Que.7	Find real and imaginary parts of $(\sqrt{i})^{\sqrt{i}}$. $\left[\mathbf{Re(z)=e^{-\left(\pi k+\frac{\pi}{4}\right)\frac{1}{\sqrt{2}}}\left[\cos\left(\pi k+\frac{\pi}{4}\right)\frac{1}{\sqrt{2}}\right]}\right]$ $\left[\mathbf{Im(z)=e^{-\left(\pi k+\frac{\pi}{4}\right)\frac{1}{\sqrt{2}}}\left[\sin\left(\pi k+\frac{\pi}{4}\right)\frac{1}{\sqrt{2}}\right]}\right]$	Dec-15
C	Que.8	Determine the modulus of following complex number.	
		1. $z=3+4i$. 2. $z=\frac{1-2i}{i-1}$ 3. $z=\frac{1-7i}{(2+i)^2}$ **[Nov-11]** $\left[\mathbf{5\ ;\ \sqrt{\frac{5}{2}}\ ;\ \sqrt{2}}\right]$	
C	Que.9	Is $\operatorname{Arg}(z_1\,z_2)=\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)$? Justify.	Jun-12
C	Que.10	Find the Principal Value of argument (Principal Argument). 1. $z=i$ 2. $z=\frac{-2}{1+i\sqrt{3}}$ **[Nov-11 ; Dec-14]** 3. $z=\sqrt{3}+i$ 4. $z=-\sqrt{3}+i$ 5. $z=-\sqrt{3}-i$ 6. $z=\sqrt{3}-i$ $\left[\mathbf{\frac{\pi}{2};\frac{2\pi}{3};\frac{\pi}{6};\frac{5\pi}{6};-\frac{5\pi}{6};-\frac{\pi}{6}}\right]$	

Basic Definition

Distance

Let $\mathrm{z = a + ib}$ and $\mathrm{w = c + id}$ be complex numbers. Distance between z & w is defined as below.

$$\mathrm{i.e.}\,|\mathrm{z - w}| = \sqrt{(\mathrm{a - c})^2 + (\mathrm{b - d})^2}$$

So, Modulus of a complex number z, $|z| = \sqrt{a^2 + b^2}$ is distance form origin.

Circle

If z' is a complex number and r is a positive number, then equation of circle is $|z - z'| = r$.

It gives the set of all those z' whose distance from z is r.[points on the boundary] [See fig A]

Open Circular Disk

The equation $|z - z'| < r$ means set of all points inside the disk of radius r about a.

Here, “ OPEN ” means that points on the boundary of circle are not in the set. [See Fig B]

Closed Circular Disk

The equation $|z - z'| \leq r$ means set of all points on the boundary and inside the disk of radius r about a. It is union of circle and open circular disk.

Here, “ CLOSED ” means that points on the boundary of circle are in the set. [See Fig C]

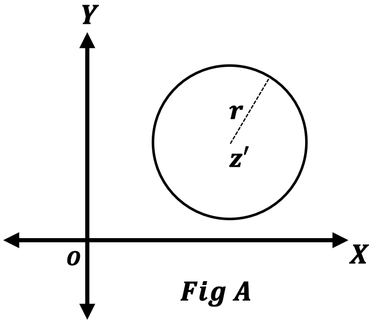


Fig A

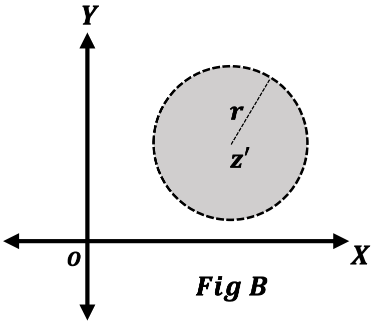


Fig B

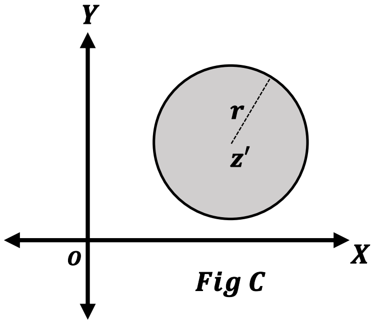


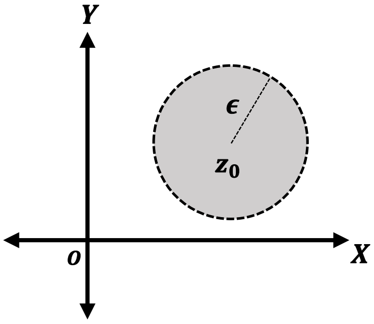
Fig C

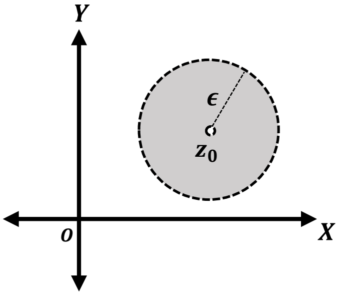
Neighborhood

The neighborhood of a point z_0 is set of points inside the circle centered at z_0 and radius ϵ.

$$\mathrm{i.e.}\,|z - z_0| < \epsilon$$

Neighborhood is nothing but a open circular disk with center z_0 and radius ϵ.



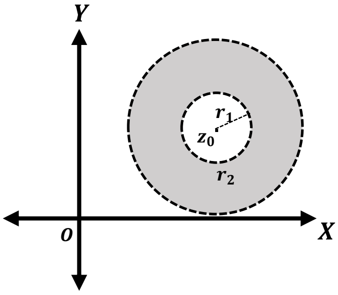


Deleted Neighborhood

The deleted neighborhood of a point z_0 is set of points inside the circle centered at z_0 and radius ϵ except the center z_0.

$$\text{i. e. } 0 < |z - z_0| < \epsilon$$

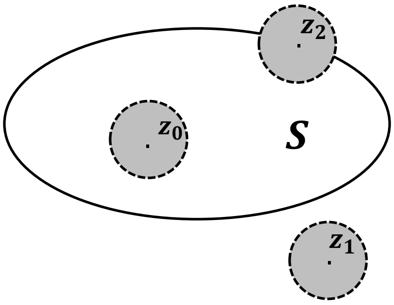
A deleted neighborhood is also known as "Punctured Disk".



Annulus OR Annular Region

The region between two concentric circle of radii r_1 & r_2 can be represented as

$$\text{i. e. } r_1 < |z - z_0| < r_2$$



Interior Point , Exterior Point and Boundary Point

A point z_0 is said to be interior point of a set S whenever there is some neighborhood of z_0 that contains only points of S.

A point z_1 is said to be exterior point of a set S whenever there is no neighborhood of z_1 that contains only points of S.

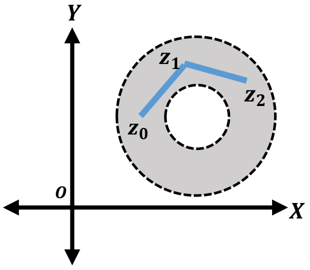
A point z_2 is said to be boundary point of a set S whenever neighborhood of z_2 contains both interior and exterior as well.

Open Set

A set is open if it contains none of the boundary points.

Closed Set

A set is said to be closed set if it contains all of the boundary points.



Connected Set

A open set S is connected if each pair of points z_0 and z_2 in it can be joined by a polygonal line, consisting of finite number of line segments joined end to end that lies entirely in S.

Domain and Region [Jun-12 ; Jun-14]

A set S is said to be domain if set S is open and connected. Note that any neighborhood is a Domain. A domain together with some, none or all of its boundary points is called region.

Compact region

A set S is said to be domain if set S is closed and connected.

Exercise-2

Que.1	Sketch the following region and check whether it is open, closed, domain, connected or bounded.	
C	1. $S = \{z \,/\, -1 < \mathrm{Im}(z) < 2\}$	Jun-12
H	2. $\mathrm{Re}\, z \geq 4$	Dec-15
H	3. $\mathrm{Im}\, z > 1$	Dec-14
C	4. $\lvert z\rvert \leq 1$	Jun-13
C	5. $\lvert z-2+i\rvert \leq 1$	Dec-13
T	6. $\lvert z-1+2i\rvert \leq 2$	Jun-12 Jun-14
C	7. $1 < \lvert z+i\rvert \leq 2$	Dec-15
T	8. $\lvert 2z+1+i\rvert < 4$	Dec-15
C	9. $0 \leq \arg z \leq \frac{\pi}{4}$	Dec-14

Formula To Find Square Root Of Complex Number

Let, $z = x + iy$ be a complex number. Formula for finding square root of z is as below,

$$\sqrt{x+iy} = \pm\left[\sqrt{\frac{|z|+x}{2}} + i(sign\ of\ y)\sqrt{\frac{|z|-x}{2}}\right]$$

Exercise-3

C	Que.1	Find $\sqrt{-8+6i}$. $[\pm\,(\mathbf{1}+\mathbf{3}\boldsymbol{i})]$	
C	Que.2	Find the roots of the equation $z^2 + 2iz + 2 - 4i = 0$. $[\boldsymbol{z} = \mathbf{1}+\boldsymbol{i}\,, -\mathbf{1}-\mathbf{3}\boldsymbol{i}]$	
H	Que.3	Solve the Equation of $z^2 - (5+i)z + 8 + i = 0$. $[\boldsymbol{z} = \mathbf{3}+\mathbf{2}\boldsymbol{i}\,, \mathbf{2}-\boldsymbol{i}]$	Jun-10
T	Que.4	Find the roots of the equation $z^2 - (3-i)z + 2 - 3i = 0$. $[\boldsymbol{z} = \mathbf{2}-\mathbf{3}\boldsymbol{i}\,, \mathbf{1}+\boldsymbol{i}]$	May-15
C	Que.5	Find the roots common to equation $z^4 + 1 = 0$ and $z^6 - i = 0$. $\left[\boldsymbol{z} = \pm\frac{\mathbf{1}-\boldsymbol{i}}{\sqrt{\mathbf{2}}}\right]$	Dec-15

Procedure To Finding Out nth Root Of a Complex Number

Let, $z = r(\cos\theta + i\sin\theta)\ ; r > 0$

For, $n \in \mathbb{N}$

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

$$= r^{\frac{1}{n}}\,\mathrm{e}^{i\left(\frac{\theta+2k\pi}{n}\right)}\ ; k = 0,1,2,\dots,n-1$$

Where, $r^{\frac{1}{n}}$ is positive nth root of r.

By putting $= 0,1,2,\dots,n-1$, we have distinct roots of $z^{\frac{1}{n}}$.

For $= n, n+1, n+2, \dots$, we have repeated roots of $z^{\frac{1}{n}}$.

Exercise-4

C	Que.1	Show that if c is any n^{th} root of Unity other than Unity itself , then $1 + c + c^2 + \cdots + c^{n-1} = 0$ **OR** Prove that the n roots of unity are in Geometric Progression.	Nov-10 Jun-14
C	Que.2	State De Moivre's formula. Find and graph all fifth root of unity in complex plane. $\left[\mathbf{z = e^{i\left(\frac{2k\pi}{5}\right)}\ ; k = 0, 1, 2, 3, 4}\right]$	Jun-13
H	Que.3	State De Moivre's formula. Find and graph all sixth root of unity in complex plane. $\left[\mathbf{z = e^{i\left(\frac{k\pi}{3}\right)}\ ; k = 0, 1, 2, 3, 4, 5}\right]$	Dec-13
T	Que.4	Find and plot the square root of $4i$. $\left[\mathbf{z = \pm\sqrt{2}(1 + i)}\right]$	
C	Que.5	State De Moivre's formula. Find and plot all root of $\sqrt[3]{8i}$. $\left[\mathbf{z = 2\ e^{i\left(\frac{4k+1}{6}\right)\pi}\ ; k = 0, 1, 2}\right]$	Jun-10 Dec-15
T	Que.6	Find and plot all the roots of $(1 + i)^{\frac{1}{3}}$. $\left[\mathbf{z = 2^{\frac{1}{6}}\ e^{i\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)}\ ; k = 0, 1, 2}\right]$	

Trigonometric (Circular) Functions Of a Complex Number

By Euler's Formula,

$e^{iz} = \cos z + i \sin z \Rightarrow e^{-iz} = \cos z - i \sin z$

- $e^{iz} + e^{-iz} = 2\cos z \Rightarrow \cos z = \frac{e^{iz}+e^{-iz}}{2}$
- $e^{iz} - e^{-iz} = 2i \sin z \Rightarrow \sin z = \frac{e^{iz}-e^{-iz}}{2i}$

Hyperbolic Function Of a Complex Number	**Relation between Circular and Hyperbolic Functions**
✓ $\cosh z = \dfrac{e^z + e^{-z}}{2}$ ✓ $\sinh z = \dfrac{e^z - e^{-z}}{2}$ ✓ $\tanh z = \dfrac{e^z - e^{-z}}{e^z + e^{-z}}$	✓ $\sin ix = i \sinh x$ ✓ $\sinh ix = i \sin x$ ✓ $\cos ix = \cosh x$ ✓ $\cosh ix = \cos x$ ✓ $\tan ix = i \tanh x$ ✓ $\tanh ix = i \tan x$
Hyperbolic Identities	**Inverse Hyperbolic Functions**
✓ $\cosh^2 x - \sinh^2 x = 1$ ✓ $\operatorname{sech}^2 x + \tanh^2 x = 1$ ✓ $\coth^2 x - \operatorname{cosech}^2 x = 1$	✓ $\sinh^{-1} z = \log\left(z + \sqrt{z^2+1}\right)$ ✓ $\cosh^{-1} z = \log\left(z + \sqrt{z^2-1}\right)$ ✓ $\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$

Show that

$\boldsymbol{\sinh^{-1} z = \log\left(z + \sqrt{z^2+1}\right), \cosh^{-1} z = \log\left(z + \sqrt{z^2-1}\right)\ \&\ \tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)}$**.**

Proof :

Let $w = \sinh^{-1} z \Rightarrow z = \sinh w = \frac{e^w - e^{-w}}{2}$

$$\Rightarrow z = \frac{e^{2w}-1}{2e^w}$$

$$\Rightarrow e^{2w} - 2ze^w - 1 = 0$$

$$\Rightarrow e^w = \frac{2z \pm \sqrt{4z^2+4}}{2} = z + \sqrt{z^2+1}$$

$$\Rightarrow w = \log\left(z + \sqrt{z^2+1}\right)$$

$$\Rightarrow \boldsymbol{\sinh^{-1} z = \log\left(z + \sqrt{z^2+1}\right) \ldots (A)}$$

Let $w = \cosh^{-1} z \Rightarrow z = \cosh w = \frac{e^w + e^{-w}}{2}$

$$z = \frac{e^{2w}+1}{2e^w}$$

$$\Rightarrow e^{2w} - 2ze^{w} + 1 = 0$$

$$\Rightarrow e^{w} = \frac{2z \pm \sqrt{4z^2 - 4}}{2} = z + \sqrt{z^2 - 1}$$

$$\Rightarrow w = \log\left(z + \sqrt{z^2 - 1}\right)$$

$$\Rightarrow \boldsymbol{\cosh^{-1} z = \log\left(z + \sqrt{z^2 - 1}\right)} \dots (\mathbf{B})$$

Let $w = \tanh^{-1} z \Rightarrow z = \tanh w = \frac{\sinh w}{\cosh w} = \frac{e^{w}-e^{-w}}{e^{w}+e^{-w}}$

$$\Rightarrow z = \frac{e^{w} - e^{-w}}{e^{w} + e^{-w}}$$

Taking componendo and dividendo, we get

$$\Rightarrow \frac{1+z}{1-z} = \frac{(e^{w} + e^{-w}) + (e^{w} - e^{-w})}{(e^{w} + e^{-w}) - (e^{w} - e^{-w})} = \frac{2e^{w}}{2e^{-w}} = e^{2w}$$

$$\Rightarrow 2w = \log\left(\frac{1+z}{1-z}\right)$$

$$\Rightarrow \boldsymbol{w = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)} \Rightarrow \boldsymbol{\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)} \dots (\mathbf{C})$$

Eqn. (A), (B) & (C) are required equations.

Exercise-5

C	**Que.1**	Prove that $\sin^{-1} z = -i\ln\left(iz + \sqrt{1 - z^2}\right)$	**Jun-13**
T	**Que.2**	Prove that $\tan^{-1} z = \frac{i}{2}\log\frac{i+z}{i-z}$.	**Jun-11**
C	**Que.3**	Show that $\cosh^{-1} z = \ln\left(z + \sqrt{z^2 - 1}\right)$	**Jun-14**
T	**Que.4**	Prove that $\operatorname{sech}^{-1} x = \log\left[\frac{1+\sqrt{1-x^2}}{x}\right]$.	**Dec-15**
C	**Que.5**	Show that $\cos(i\bar{z}) = \overline{\cos(\imath z)}$ for all z.	**Nov-12**
T	**Que.6**	Show that $\overline{\sin(\imath z)} = \sin(i\bar{z})$ if and only if $z = n\pi i$ $(n \in Z)$.	**Dec-14**
C	**Que.7**	Expand $\cosh(z_1 + z_2)$.	**Nov-12**
H	**Que.8**	Expand $\sinh(z_1 + z_2)$.	
C	**Que.9**	Prove that $\lvert e^{-2z} \rvert < 1$ if and only if $\mathrm{Re}(z) > 0$.	**Nov-12**
C	**Que.10**	Find all Solution of $\sin z = 2$.	**Jun-10**

Logarithm of a complex number

Polar representation of complex number, $z = re^{i\theta}$

$$\Rightarrow z = re^{i(\theta + 2k\pi)}$$

$$\Rightarrow \log z = \ln r + i(\theta + 2k\pi)$$

$\Rightarrow \log z = \ln\left(\sqrt{x^2+y^2}\right) + i\left(2k\pi + \tan^{-1}\left(\frac{y}{x}\right)\right)$; $k = 0, \pm 1, \pm 2, \ldots$

is called “GENERAL VALUE OF LOGARITHM”.

If $k = 0$,

$\Rightarrow \mathrm{Log}\, z = \ln\left(\sqrt{x^2+y^2}\right) + i\tan^{-1}\left(\frac{y}{x}\right)$ is called “PRINCIPAL VALUE OF LOGARITHM”.

Note

✓ In Complex analysis,
  - Log is used for Complex Single-Valued Function.
  - log is used for Complex Multi-Valued Function.
  - ln is used for Real Valued Function.

Exercise-6

C	Que.1	Define $\log(x+iy)$. Determine $\log(1-i)$. $\left[\ln\sqrt{2} - \frac{i\pi}{4}\right]$	Jun-12
C	Que.2	Show that the set of values of $\log(i^2)$ is not the same as the set of values $2\log i$.	Nov-12
H	Que.3	For the principle branch show that $\mathrm{Log}(i^3) \neq 3\mathrm{Log}(i)$.	Jun-13
T	Que.4	Find the principal value of $\left[\frac{e}{2}\left(-1-i\sqrt{3}\right)\right]^{3\pi i}$. $\left[3\pi i\left(1-\frac{2\pi i}{3}\right)\right]$	Nov-11
C	Que.5	Find all root s of the Equation $\log z = \frac{i\pi}{2}$. $[z = i]$	Nov-12 Dec-15
H	Que.6	Prove that $i^i = e^{-(4n+1)\frac{\pi}{2}}$.	Jun-14
C	Que.7	Find the value of $(-i)^i$. $\left[e^{-(4n-1)\frac{\pi}{2}}\right]$	Dec-13

Function of a Complex Variable

If corresponding to each value of a complex variable $z = x + iy$ in a given region R, there correspond one or more values of another complex variable $w = u + iv$ then, *w* is called a function of the complex variable *z* and is denoted by

$$w = f(z) = u + iv$$

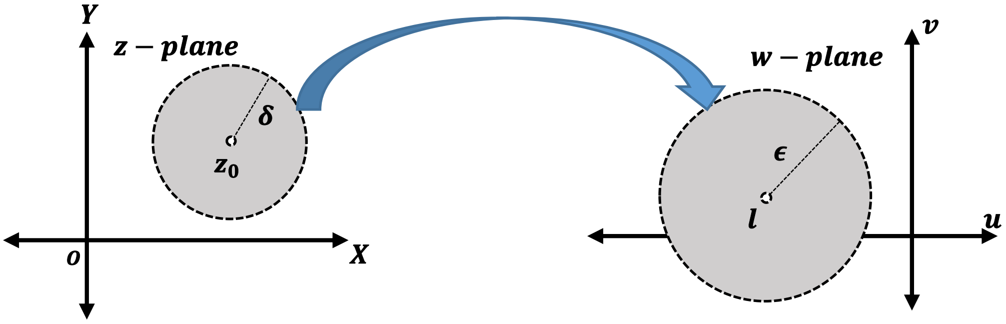
Where, u and v are the real and imaginary part of w respectively and u and v are function of real variable x and y.

$$\text{i.e. } w = f(z) = u(x,y) + i\,v(x,y)$$

Limit Of Complex Function

A function $f(z)$ is said to have a limit l, if for each +ve number ϵ, there is +ve number δ such that

$$\text{i.e. } |f(z) - l| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta$$



Continuity of Complex function

A complex valued function f(z) is said to be continuous at a point $z = z_0$ if

1. $f(z_0)$ exists
2. $\lim_{z\to z_0} f(z)$ exist
3. $\lim_{z\to z_0} f(z) = f(z_0)$

Remark

- ✓ $f(z) = u(x,y) + i\,v(x,y)$ is continuous iff $u(x,y)$ and $v(x,y)$ are continuous.
- ✓ If any one of these three conditions of continuity is not satisfied then $f(z)$ is discontinuous at $z = z_0$.

Differentiability of complex function

Let $w = f(z)$ be a continuous function and z_0 be a fixed point then f(z) is said to be differentiable at z_0 if $\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists, then the derivative of f(z) at z_0 is denoted by $f'(z_0)$ and is defined as

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Remark

- ✓ The rules of differentiation are same as in calculus of real variables.
- ✓ If function is differentiable, then it is continuous.

Exercise-7

C	Que.1	Prove $\lim_{z\to 1} \frac{iz}{3} = \frac{i}{3}$ by definition.	Jun-12
H	Que.2	Using the definition of limit, show that if $f(z) = iz$ in the open disk $\lvert z\rvert < 1$, then $\lim_{z\to 1} f(z) = i$.	Dec-14
C	Que.3	Show that the limit of the function does not exist $f(z) = \begin{cases} \frac{Im(z)}{\lvert z\rvert}, & z \neq 0 \\ 0, & z = 0 \end{cases}$	
C	Que.4	Discuss the continuity of $f(z) = \begin{cases} \frac{\bar{z}}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$ at origin.	May-15

T	**Que.5**	Discuss continuity of $f(z) = \begin{cases} \frac{Re(z^2)}{\lvert z\rvert^2} ; z \neq 0 \\ 0 ; z = 0 \end{cases}$ at $z = 0$.	**Dec-15**
H	**Que.6**	Find out and give reason weather f(z) is continuous at $z = 0$,if $f(z) = \begin{cases} \frac{Re(z^2)}{\lvert z\rvert}, & z \neq 0 \\ 0 \quad , & z = 0 \end{cases}$	**Jun-10**
C	**Que.7**	Find the derivative of $\frac{z-i}{z+i}$ at i.	**Jun-10**
C	**Que.8**	Discuss the differentiability of $f(z) = x^2 + iy^2$.	**May-15**
H	**Que.9**	Show that $f(z) = z\,Im(z)$ is differentiable only at $z = 0$ and $f'(0) = 0$.	**Nov-11**
C	**Que.10**	Show that $f(z) = \lvert z\rvert^2$ is continuous at each point in the plane, but not differentiable.	**Dec-14**
T	**Que.11**	Show that $f(z) = \bar{z}$ is nowhere differentiable.	**Jun-14**

Analytic Function

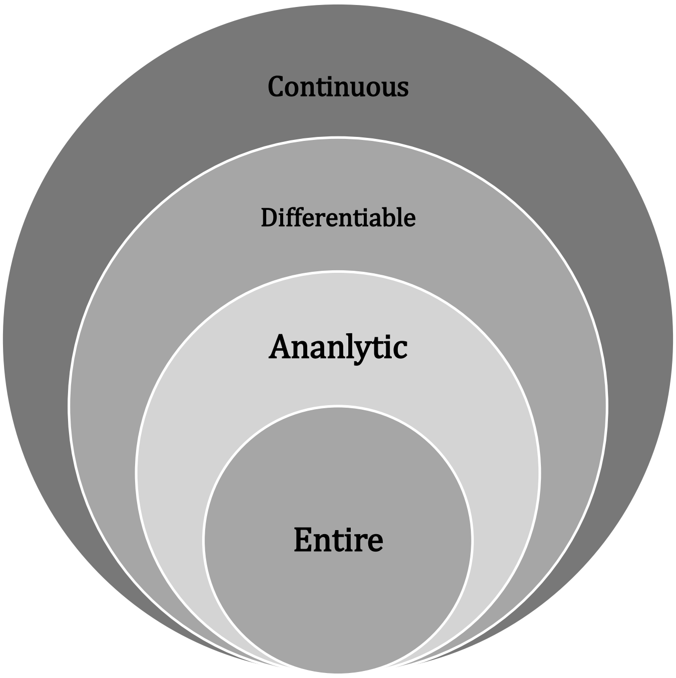
A function f(z) is said to be analytic at point $z_0 = x_0 + iy_0$ if the function is differentiable at point z_0 as well as it is differentiable everywhere in the neighbourhood of z_0.

Examples :

1. $f(z) = \frac{1}{z}$ is analytic at each non-zero point in the finite complex plane.
2. $f(z) = \lvert z\rvert^2$ is not analytic at any non-zero point because it is not differentiable at any non-zero complex number.
3. $f(z) = \bar{z}$ is nowhere analytic because it is nowhere differentiable.

Remark

- ✓ Analytic functions are also known as regular or holomorphic functions.
- ✓ A function f is analytic everywhere in domain D iff it is analytic at each point of domain D.
- ✓ A function f is analytic everywhere in domain D then f is known as entire function in D.



Cauchy-Riemann Equations[C-R equation]

If $\mathrm{u(x, y)}$ and $\mathrm{v(x, y)}$ are real single-valued functions of x ad y such that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous in the region R,then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \frac{\partial u}{\partial y} = -\ \frac{\partial v}{\partial x}$$

are known as Cauchy-Riemann Equations.

Necessary and Sufficient Conditions For $f(z)$ to be Analytic

The necessary and sufficient conditions for the function $\mathrm{f(z) = u(x, y) + iv(x, y)}$ to be analytic in a region R are

- ✓ $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous functions of x and y in the region R.
- ✓ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\ \frac{\partial v}{\partial x}$

i.e. Cauchy-Riemann equations are satisfied.

Proof: (Necessary Condition)

Let, $\mathrm{f(z) = u(x, y) + iv(x, y)}$ is analytic in a region $\mathbb{R}$.

$\Longrightarrow$ f(z) is differentiable at every point of the region $\mathbb{R}$.

$$\Longrightarrow f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \ldots (1)$$

We know that $z_0 = x_0 + iy_0$ & $\Delta z = \Delta x + i\Delta y$

Now, $z_0 + \Delta z = x_0 + \Delta x + i(y_0 + \Delta y)$

$\Longrightarrow f(z_0 + \Delta z) = u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y)$

By Eqn (1),

$$f'(z_0) = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{u(x_0+\Delta x, y_0+\Delta y) + iv(x_0+\Delta x, y_0+\Delta y) - u(x_0,y_0) - iv(x_0,y_0)}{\Delta x + i\Delta y}$$

$$= \left[\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y}\right] + i\left[\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y}\right]$$

Here, we consider two paths:

Path I: First $\Delta y \to 0$ then $\Delta x \to 0$

$$f'(z_0) = \lim_{\Delta x\to 0} \frac{u(x_0+\Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i \lim_{\Delta x\to 0} \frac{v(x_0+\Delta x, y_0) - v(x_0,y_0)}{\Delta x}$$

$$\Rightarrow f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \ldots (2)$$

Path II: First $\Delta x \to 0$ then $\Delta y \to 0$

$$\Rightarrow f'(z_0) = \lim_{\Delta y\to 0} \frac{u(x_0, y_0+\Delta y) - u(x_0,y_0)}{i\Delta y} + i \lim_{\Delta y\to 0} \frac{v(x_0, y_0+\Delta y) - v(x_0,y_0)}{i\Delta y}$$

$$\Rightarrow f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \ldots (3)$$

Since, $f'(z_0)$ exists.So, equations (2) and (3) must be equal.

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Comparing real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, C-R equations are satisfied.

Remark

- ✓ C.R. equations are necessary condition for differentiability but not sufficient.
- ✓ If $f(z) = u(x,y) + iv(x,y)$ is an analytic function, then $u(x,y)$ and $v(x,y)$ are conjugate functions.
- ✓ If a function is differentiable $\Rightarrow$ function satisfies C.R. equation.If a function does not satisfies C.R. equation $\Rightarrow$ function is not differentiable.
- ✓ If function is differentiable at point (x_0, y_0) then derivative at z_0 is given by
  - ○ $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$. (Cartesian form)
  - ○ $f'(z_0) = e^{-i\theta}(u_r(r,\theta) + iv_r(r,\theta))$. (polar form)

Cauchy-Riemann Equations in Polar Form

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Exercise-7

C	**Que.1**	State necessary and sufficient Condition for function to be analytic and prove that necessary condition.	**Nov-10**
C	**Que.2**	The function $f(z)=\begin{cases}\frac{\bar{z}^2}{z}; & z\neq 0\\ 0; & z=0\end{cases}$ satisfies C-R equation at the origin but $f'(0)=0$ fails to exist.	**Nov-12**
	Que.3	Show that for the function $f(z)=\begin{cases}\frac{\bar{z}^2}{z}; z\neq 0\\ 0; z=0\end{cases}$ is not differentiable at $z=0$ even though Cauchy Reimann equation are satisfied at $z=0$.	**Dec-15**
C	**Que.4**	Check Whether $f(z)=\bar{z}$ is analytic or not. **[Nowhere analytic]**	**Nov-10**
H	**Que.5**	Check Whether $f(z)=2x+ixy^2$ is analytic or not at any point. **[Nowhere analytic]**	**Jun-10**
H	**Que.6**	State the necessary condition for f(z) to be analytic. For what values of z is the function $f(z)=3x^2+iy^2$ analytic ? **[Except the line $y=3x$ function is nowhere analytic]**	**Dec-15**
T	**Que.7**	Check Whether $f(z)=e^{\bar{z}}$ is analytic or not at any point. **[Nowhere analytic]**	**Jun-10**
C	**Que.8**	Is $f(z)=\sqrt{r}e^{\frac{i\theta}{2}}$ analytic? $(r>0,-\pi<\theta<\pi)$ **[Analytic except $(0,0)$]**	**Dec-15**
C	**Que.9**	Let $f(z)=z^n=r^n e^{in\theta}$ for integer n.Verify C-R equation and find its derivative. $[\mathbf{f(z)=n\,z^{n-1}}]$	**Dec-15**
H	**Que.10**	What is an analytic function? Show that $f(z)=z^3$ is analytic everywhere. **[Analytic everywhere]**	**Jun-14**
H	**Que.11**	Check Whether $f(z)=z^{\frac{5}{2}}$ is analytic or not. **[Analytic everywhere]**	**Nov-10**
C	**Que.12**	Check Whether the function $f(z)=\sin z$ is analytic or not. If analytic find it's derivative. **[Analytic everywhere]**	**Nov-11**
T	**Que.13**	Examine the analyticity of $\sinh z$. **[Nowhere analytic]**	**Dec-14**
C	**Que.14**	Show that if f(z) is analytic in a domain D & $\lvert f(z)\rvert=k(\text{cons.})$ in D then show that $f(z)=k(\text{cons.})$ in D.	**Jun-10**

C	Que.15	Let a function f(z) be analytic in a domain D prove that f(z) must be constant in D in each of following cases. 1. If f(z) is real valued for all z in D. 2. If $\overline{f(z)}$ is analytic in D.	Nov-12
C	Que.16	If f(z) is analytic function, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\lvert f'(z)\rvert = 0$.	Dec-14
C	Que.17	If f(z) is analytic function, prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left\lvert \mathrm{Re}\big(f(z)\big)\right\rvert^2 = 2\lvert f'(z)\rvert^2$.	May-15

Harmonic Functions

A real valued function $\phi(x, y)$ is said to be harmonic function in domain D if

- ✓ $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$. (Laplace Equation)
- ✓ All second order partial derivative $\phi_{xx}, \phi_{xy}, \phi_{yx}, \phi_{yy}$ are continuous.

Theorem

If $f(z) = u + iv$ is analytic in domain D then u and v are harmonic function in D.

Harmonic Conjugate

Let $u(x, y)$ and $v(x, y)$ are harmonic function and they satisfy C.R. equations in certain domain D then $v(x, y)$ is harmonic conjugate of $u(x, y)$.

Theorem

If $f(z) = u + iv$ is analytic in D iff $v(x, y)$ is harmonic conjugate of $u(x, y)$.

Remark

- ✓ If $f(z) = u + iv$ is analytic function then $v(x, y)$ is harmonic conjugate of $u(x, y)$ but $u(x, y)$ is not harmonic conjugate of $v(x, y)$. $-u(x, y)$ is harmonic conjugate of $v(x, y)$.

Milne-Thomson’s Method

This method determines the analytic function f(z) when either u or v is given.

We know that $z = x + iy$ and $\bar{z} = x - iy$

$$\therefore x = \frac{z+\bar{z}}{2} \ \& \ y = \frac{z-\bar{z}}{2i}$$

Now, $f(z) = u(x, y) + iv(x, y) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$

Putting $\bar{z} = z$, we get

$$f(z) = u(z, 0) + iv(z, 0)$$

Which is same as $f(z) = u(x, y) + iv(x, y)$ if we replace x by z and y by 0.

Now, $f(z) = u + iv$

$$\Rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{(By C. R. equations)}$$

Replacing x by z and y by 0,we get

$$f'(z) = u_x(z, 0) + i\, u_y(z, 0)$$

Integrating both the sides, with respect to z, we get

$$f(z) = \int u_x(z, 0)\, dz + \int i\, u_y(z, 0)\ dz.$$

Exercise-8

C	**Que.1**	Define: Harmonic Function. Show that $u(x, y) = x^2 - y^2$ is harmonic. Find the corresponding analytic function $f(z) = u(x, y) + iv(x, y)$. $[\mathbf{f(z) = z^2 + c}]$	**Jun-13** **Dec-15**
H	**Que.2**	Define: Harmonic Function. Show that is $u(x, y) = x^2 - y^2 + x$ harmonic.Find the corresponding analytic function $f(z) = u(x, y) + iv(x, y)$. $[\mathbf{f(z) = z^2 + z + c}]$	**Dec-13**
H	**Que.3**	Find analytic function $f(z) = u + iv$ if $u = x^3 - 3xy^2$. $[\mathbf{f(z) = z^3 + c}]$	**Jun-14**
H	**Que.4**	Show that $u(x, y) = 2x - x^3 + 3xy^2$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$. $[\mathbf{v(x, y) = 2y - 3x^2y + y^3 + c}]$	**Jun-11**
C	**Que.5**	Determine a and b such that $u = ax^3 + bxy$ is harmonic and find Conjugate harmonic. $[\mathbf{a = 0}\ ; \mathbf{b} \in \mathbb{C}]$	**Nov-10**
T	**Que.6**	Define: Harmonic Function. Show that $u = \frac{x}{x^2+y^2}$ is harmonic function for $\mathbb{R}^2 - (0,0)$.	**Jun-14**
C	**Que.7**	Define: Harmonic Function. Show that $u = x \sin x \cosh y - y \cos x \sinh y$ is harmonic.	**Jun-12**
T	**Que.8**	Define Harmonic Function. Show that the function $u(x, y) = e^x \cos y$ is harmonic. Determine its harmonic conjugate $v(x, y)$ and the analytic function $f(z) = u + iv$. $[\mathbf{v(x, y) = e^x \sin y\ \ ;\ \ f(z) = e^x \cos y + i\, e^x \sin y}\ ]$	**Dec-15**
C	**Que.9**	Determine the analytic function whose real part is $e^x(x \cos y - y \sin y)$. $[\mathbf{f(z) = z\, e^z} + \boldsymbol{c}]$	
H	**Que.10**	Determine the analytic function whose real part is $e^{2x}(x \cos 2y - y \sin 2y)$. $[\mathbf{f(z) = 4z\, e^{2z} - 6\, e^{2z}} + \boldsymbol{c}]$	
T	**Que.11**	Show that $u(x, y) = e^{x^2-y^2} \cos(2xy)$ is harmonic everywhere. Also find a conjugate harmonic for $u(x, y)$. $\left[\mathbf{v(x, y) = e^{x^2-y^2} \sin(2xy)}\right]$	**Nov-11**

C	**Que.12**	Find the analytic function $f(z) = u + iv$, if $u - v = e^x(\cos y - \sin y)$ $[\mathbf{f(z) = e^z} + \boldsymbol{c}]$	**May-15**
T	**Que.13**	Find the all analytic function $f(z) = u + iv$, if $u - v = (x - y)(x^2 + 4xy + y^2)$. $[\mathbf{f(z) = -i\,z^3} + \boldsymbol{c}]$	**Nov-12**

Introduction

Integrals of complex valued function of a complex variable are defined on curves in the complex plane, rather than on interval of real line.

Continuous arc

The set of points (x, y) defined by $x = f(t)$, $y = g(t)$, with parameter t in the interval (a, b), define a continuous arc provided f and g are continuous functions.

Smooth arc

If f and g are differentiable on arc $a \leq t \leq b$ and non-zero on open interval $a < t < b$ is called smooth arc.

Simple Curve/Simple arc/Jordan arc

A curve which does not intersect with itself. i.e. if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$.

Simple Closed Curve

A simple curve C except for the fact $z(b) \neq z(a)$; where a & b are end points of interval.

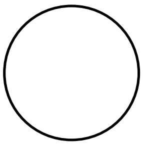


Fig. A

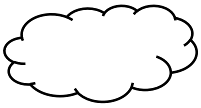


Fig. B

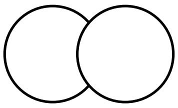


Fig. C

Contour

A contour or piecewise smooth arc, is an arc consisting of a finite number of smooth arcs join end to end.

If only initial and final values are same, a contour is called **Simple closed contour**.

A Simply connected domain D is simple closed path in D encloses only points of D.

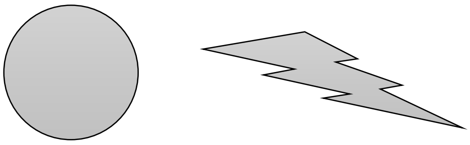
Examples

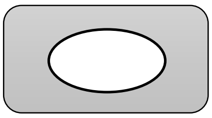
- ✓ A open disk, ellipse or any simple closed curve.

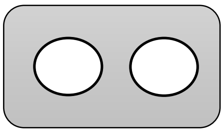
A domain that is not simply connected is called **multiply connected**.

Examples

- ✓ An annulus is multiply connected.


Simply connected


Doubly Connected


Triply Connected

Line Integral in Complex Plane

A line integral of a complex function f(z) along the curve C is denoted by $\int_c f(z)\ dz$.

Note that, if C is closed path, then line integral of f(z) is denoted by $\oint_C f(z)\ dz$.

$\oint_C f(z)\ dz$ is also known as Contour integral.

Properties of Line Integral

Linearity

$$\int_C [k_1 f(z) + k_2 g(z)]\, dz = k_1 \int_C f(z)\, dz + k_2 \int_C g(z)\, dz$$

Reversing the sense of integration

$$\int_a^b f(z)\, dz = -\int_b^a f(z)\, dz$$

Partition of Path

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz \ ; \text{where } c = c_1 \cup c_2$$

For the complex integral

$$\left|\int_C f(z)\, dz\right| \le \int_C |f(z)|\, |dz|$$

ML inequality

If $f(z)$ is continuous on a contour C, then $\left|\int_C f(z)dz\right| \le ML$.

Where $|f(z)| \le M, z \in \mathbb{C}$ and L is the length of the curve (contour)C.

Note

Real definite integrals are interpreted as area, no such interpretation

Exercise-1

C	Que.1	Evaluate $\int_0^{2+i} z^2 dz$ along the line $y = \frac{x}{2}$. $\left[\frac{2}{3} + \frac{11}{3}\mathbf{i}\right]$	Jun-13
C	Que.2	Evaluate $\int_c (x^2 - iy^2)dz$,along the parabola $y = 2x^2$ from (1,2) to (2,8). $\left[\frac{511}{3} - \frac{49}{5}\mathbf{i}\right]$	Jun-12 Dec-15
C	Que.3	Evaluate $\int_c \text{Re}(z)dz$ where C is the shortest path from $(1 + i)$ to $(3 + 2i)$. $[\mathbf{4 + 2i}]$	Jun-14
H	Que.4	Evaluate $\int_c \bar{z}\, dz$ from $z = 1 - i$ to $z = 3 + 2i$ along the straight line. $\left[\frac{11}{2} + \mathbf{5i}\right]$	Jun-12
H	Que.5	Evaluate $\int_c z^2 dz$ where C is line joins point (0,0) to (4,2). $\left[\frac{16}{3} + \frac{88}{3}\mathbf{i}\right]$	Dec-13
C	Que.6	Evaluate $\int_c (x - y + ix^2)dz$,Where c is a straight line from $z = 0$ to $z = 1 + i$. $\left[\frac{\mathbf{i(1+i)}}{3}\right]$	
C	Que.7	Evaluate $\int_0^{4+2i} \bar{z}\, dz$ along the curve $z = t^2 + it$. $\left[\mathbf{10} - \frac{8}{3}\mathbf{i}\right]$	Dec-11

C	Que.8	Evaluate $\int_c (x-y+ix^2)dz$,Where c is along the imaginary axis from $z=0$ to $z=1, z=1$ to $z=1+i$ & $z=1+i$ to $z=0$. $\left[\mathbf{\frac{3i-1}{6}}\right]$	
H	Que.9	Evaluate $\int_c \mathrm{Re}(z)dz$,Where c is a straight line from (1,1) to (3,1) & then from (3,1) to (3,2). $[\mathbf{4+3i}]$	
H	Que.10	Evaluate $\int_c \bar{z}\,dz$, where C is along the sides of triangle having vertices $z=0,1,i$. $[\mathbf{i}]$	May-15
C	Que.11	Evaluate $\int_c z^2dz$,Where c is the path joining the points $1+i$ and $2+4i$ along (a) the parabola $x^2=y$ (b) the curve $x=t, y=t^2$. $\left[-\mathbf{\frac{86}{3}}-\mathbf{6i}\right]$	
H	Que.12	Evaluate $\int_c \mathrm{Re}(z^2)dz$.Where c is the boundary of the square with vertices $0, i, 1+i, 1$ in the clockwise direction. $[\mathbf{-1-i}]$	
C	Que.13	Evaluate $\int_c f(z)dz$. Where f(z) is defined by $f(z)=\begin{cases}1 & ;y<0\\ 4y & ;y>0\end{cases}$. c is the arc from $z=-1-i$ to $z=1+i$ along the curve $y=x^3$. $[\mathbf{2+3i}]$	
C	Que.14	Find the value of integral $\int_c \bar{z}\,dz$ where c is the right-hand half $z=2e^{i\theta}$; $(-\frac{\pi}{2}\le\theta\le\frac{\pi}{2})$ of the circle $\|z\|=2$, from $z=-2i$ to $z=2i$. $[\mathbf{4\pi i}]$	Dec-14

Maximum Modulus Theorem

If f(z) is analytic inside and on a simple closed curve C, then maximum value of |f(z)| occurs on C, unless f(z) must be constant.

Exercise-2

C	Que.1	Find an upper bound for the absolute value of the integral $\int_c e^z\,dz$, where c is the line segment joining the points (0,0) and $(1,2\sqrt{2})$. $[\mathbf{3e}]$	Jun-10
C	Que.2	Without using integration, show that $\left\|\oint_C \frac{e^z}{z+1}dz\right\|\le\frac{8\pi e^4}{3}$; C: $\|z\|=4$.	Jun-13
H	Que.3	Find an upper bound for the absolute value of the integral $\int_c \frac{dz}{z^2+1}$, where c is the arc of a circle $\|z\|=2$ that lies in the first quadrant. $\left[\mathbf{\frac{\pi}{3}}\right]$	
T	Que.4	Find an upper bound for the absolute value of the integral $\int_c z^2dz$, where c is the straight line segment from 0 to $1+i$. $[\mathbf{2\sqrt{2}}]$	Jun-14

Cauchy's Integral Theorem (Cauchy Goursat's Theorem) (Jun-'13)

If f(z) is an analytic function in a simply connected domain D and $f'(z)$ is continuous at each point within and on a simple closed curve C in D, then

$$\oint_C f(z)dz = 0$$

Liouville's Theorem

If f(z) is an analytic and bounded function for all z in the entire complex plane, then f(z) is constant.

Exercise-3

C	**Que.1**	State and Prove Cauchy integral theorem.	**Jun-10** **Dec-14**
C	**Que.2**	Evaluate $\oint_C e^{z^2} dz$, where C is any closed contour. Justify your answer. [**0**]	**Dec-13**
H	**Que.3**	If C is any simple closed contour, in either direction, then show that $\int_c \exp(z^3)\ dz = 0$. [**0**]	**Dec-14**
C	**Que.4**	Evaluate $\oint_C (z^2 + 3)dz$, where C is any closed contour. Justify your answer. [**0**]	**Jun-13**
H	**Que.5**	Evaluate $\oint_C (z^2 - 2z - 3)dz$, where C is the circle $\|z\| = 2$. [**0**]	
C	**Que.6**	Evaluate $\int_c \frac{dz}{z^2}$, c is along a unit circle. [**0**]	**May-15**
H	**Que.7**	Evaluate $\oint_C \frac{z}{z-3} dz$, where C is the unit circle $\|z\| = 1$. [**0**]	
T	**Que.8**	Evaluate $\oint_C \frac{e^z}{z+i} dz$, where C is the unit circle $\|z - 1\| = 1$. [**0**]	**May-15**
C	**Que.9**	Evaluate $\oint_C \frac{z+4}{z^2+2z+5} dz$, where C is the circle $\|z + 1\| = 1$. [**0**]	

Cauchy's Integral Formula (Dec-'15)

If f(z) is an analytic within and on a simple closed curve C and z_0 is any point interior to C, then

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i \, f(z_0)$$

the integration being taken counterclockwise.

$$\text{In general, } \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0)$$

Exercise-4

C	Que.1	Prove that $\int_c \frac{dz}{z-a} = 2\pi i . \int_c (z-a)^n dz = 0$ $[n \in \mathbb{Z} - \{-1\}]$, where C is the circle $\lvert z - a\rvert = r$.	Jun-14
H	Que.2	Evaluate $\oint_C \frac{z^2-4z+4}{(z+i)} dz$, where C is $\lvert z\rvert = 2$. $[(\mathbf{-8+6i})\boldsymbol{\pi}]$	Dec -13
C	Que.3	Evaluate $\oint_C \frac{\sin 3z}{z+\frac{\pi}{2}} dz$, where C is the circle is $\lvert z\rvert = 5$. $[\mathbf{2\pi i}]$	Dec -11
C	Que.4	Evaluate $\oint_C \frac{5z+7}{z^2+2z-3} dz$, where C is $\lvert z - 2\rvert = 2$. $[\mathbf{6\pi i}]$	Jun-13
C	Que.5	Evaluate $\oint_C \frac{dz}{z^2+1}$, where C is $\lvert z + i\rvert = 1$, counterclockwise. $[\mathbf{-\pi}]$	Jun-10 Jun-14
H	Que.6	Evaluate $\oint_C \frac{2z+6}{z^2+4} dz$, where C is $\lvert z - i\rvert = 2$. $[(\mathbf{3+2i})\boldsymbol{\pi}]$	Jun-13
T	Que.7	Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)} dz$, where C is $\lvert z\rvert = 2$. $\left[-\frac{\pi}{2}\mathbf{i}\right]$	Dec -13
C	Que.8	Evaluate $\int_c \frac{z}{z^2+1} dz$, where c is the circle (i) $\left\lvert z + \frac{1}{2}\right\rvert = 2$ (ii)$\lvert z + i\rvert = 1$. $[\mathbf{2\pi i, -\pi i}]$	Dec-15
H	Que.9	Evaluate $\int_c \frac{1+z^2}{1-z^2}$ dz, where *c* is unit circle centred at (1) $z = -1$ (2) $z = i$. $[\mathbf{2\pi i, 0}]$	Dec-15
C	Que.10	State Cauchy-Integral theorem. Evaluate $\oint_C \left(\frac{3}{z-i} - \frac{6}{(z-i)^2}\right) dz$, where C: $\lvert z\rvert = 2$. $[\mathbf{6\pi i}]$	May-15
H	Que.11	Evaluate $\int_{C:\lvert z\rvert=2} \frac{dz}{z^3(z+4)}$. $\left[\frac{\pi i}{32}\right]$	Dec-15
T	Que.12	Evaluate $\oint_C \frac{e^z}{z(1-z)^3} dz$, where C is (a)$\lvert z\rvert = \frac{1}{2}$ (b)$\lvert z - 1\rvert = \frac{1}{2}$. $[\mathbf{2\pi i, -\pi i e}]$	Jun-10

Power Series

A series of the for

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + a_n(z-z_0)^n + \cdots$$

In which the coefficients $a_1, a_2, a_3, \ldots., a_n, ..$ are real or complex and z_0 is a fixed point in the complex z-plane is called a Power series in powers of $(z - z_0)$ or about z_0 or a power series centered at z_0.

Convergence of a power series in a disk

The series converges everywhere inside a circular disk $|z - z_0| < R$ and diverges everywhere outside the disk $|z - z_0| > R$.

Here, R is called the radius of convergence and the circle $|z - z_0| = R$ is called the circle of convergence.

Radius of Convergence

Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series. Radius of convergence R for power series is defined as below

$$\mathbf{R} = \lim_{\mathbf{n} \to \infty} \left|\frac{\mathbf{a_n}}{\mathbf{a_{n+1}}}\right| \quad \text{or} \quad \mathbf{R} = \lim_{\mathbf{n} \to \infty} |\mathbf{a_n}|^{-\frac{1}{n}}$$

Exercise-1

C	**Que.1**	Discuss the convergence of $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2}(z-3i)^n$. Also find the radius of convergence. $\left[\mathbf{R} = \frac{1}{4}\right]$	**Dec-10** **Dec-15**
T	**Que.2**	Find the radius of convergence of the $\sum_{n=0}^{\infty}(n+2i)^n z^n$. $[\mathbf{R} = \mathbf{0}]$	**Jun-10**
C	**Que.3**	Find the radius of convergence of $\sum_{n=1}^{\infty}\left(\frac{6n+1}{2n+5}\right)^2 (z-2i)^n$. $[\mathbf{R} = \mathbf{1}]$	**May-15**
H	**Que.4**	Find the radius of convergence of the power series a) $\sum_{n=0}^{\infty} \frac{2^n}{n!} z^n$ b) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2} z^n$ $\left[\mathbf{R} = \infty, \frac{\mathbf{1}}{\mathbf{e}}\right]$	

Taylor’s series

Let f(z) be analytic everywhere inside a circle C with centre at z_0 and radius R. then at each point Z inside C,we have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^n(z_0)}{n!}(z - z_0)^n + \cdots$$

Maclaurin’s Series

If we take $z_0 = 0$, the above series reduces to

$$f(z) = f(0) + f'(0)z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^n(0)}{n!}z^n + \cdots$$

Geometric Series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1) \qquad \frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \quad (|z| < 1)$$

Laurent’s Series

If f(z) is analytic within and on the ring (annulus) shaped region R bounded by two concentric circles C_1 and C_2 od radii R_1 and R_2 $(R_2 < R_1)$ resp. having center at the point $z = z_0$, then for all z in R, f(z) is uniquely represented by a convergent Laurent’s series given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

Where,

$$a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(t)}{(t - z_0)^{n+1}}\,dt \quad \& \quad a_{-n} = \frac{1}{2\pi i}\int_{C_2} \frac{f(t)}{(t - z_0)^{-n+1}}\,dt$$

Here, $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$ is known Pricipal Part of Laurent’s series.

Exercise-2

C	Que.1	Derive the Taylor’s series representation in $\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}$; where $\lvert z - i\rvert < \sqrt{2}$	**Dec-12**
C	Que.2	Obtain the Taylor’s series $f(z) = \sin z$ in power of $\left(z - \frac{\pi}{4}\right)$.	**Dec-15**

H	Que.3	Develop $f(z) = \sin^2 z$ in a Maclaurin series and find the radius of convergence.	Jun-10
H	Que.4	Find Maclaurin series representation of $f(z) = \sin z$ in the region $\|z\| < \infty$.	Dec-11
C	Que.5	Find the Laurent's expansion of $\frac{\sin z}{z^3}$ at $z = 0$ and classify the singular point $z = 0$.	Dec-15
H	Que.6	Expand $f(z) = \frac{1-e^z}{z}$ in Laurent's series about $z = 0$ and identify the singularity.	Jun-10
C	Que.7	Show that when $0 < \|z-1\| < 2$, $\frac{z}{(z-1)(z-3)} = \frac{-1}{2(z-1)} - 3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}$	Dec-12
C	Que.8	Find the Laurent's expansion in power of z that represent $f(z) = \frac{1}{z(z-1)}$ for domain (a) $0 < \|z\| < 1$ (b) $0 < \|z-1\| < 1$.	Dec-13
H	Que.9	Find the Laurent's expansion of $f(z) = \frac{7z-2}{(z+1)z(z-2)}$ in the region $1 < \|z+1\| < 3$.	Jun-11
T	Que.10	Write the two Laurent series expansion in powers of z that represent the function $f(z) = \frac{1}{z^2(1-z)}$ in certain domains, also specify domains.	Dec-10 Jun-13
C	Que.11	Expand $f(z) = \frac{1}{(z+1)(z-2)}$ in the region (i)$\|z\| < 1$ (ii)$1 < \|z\| < 2$ (iii) $\|z\| > 2$.	May-15
H	Que.12	Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in the region (i)$\|z\| < 1$ (ii)$1 < \|z\| < 2$.	Jun-14
H	Que.13	Expand $f(z) = -\frac{1}{(z-1)(z-2)}$ in the region (a) $\|z\| < 1$ (b) $1 < \|z\| < 2$ (c) $\|z\| > 2$	Dec-10 Dec-14
H	Que.14	Expand $f(z) = \frac{1}{(z+2)(z+4)}$ for the region (a) $\|z\| < 2$ (b) $2 < \|z\| < 4$ (c) $\|z\| > 4$	Jun-12
C	Que.15	Expand $\frac{1}{z(z^2-3z+2)}$ in a Laurent series about $z = 0$ for the regions (a) $0 < \|z\| < 1$ (b) $\|z\| > 2$	Dec-15

H	Que.16	Expand $f(z)=\frac{1}{(z+1)(z+3)}$ in Laurent's series in the interval $1<\lvert z\rvert<3$	Dec-11
C	Que.17	Find the series of $f(z)=\frac{z}{(z-1)(z-4)}$ in terms of $(z+3)$ valid for $\lvert z+3\rvert<4$	Jun-12

Definition

Singular point

A point z_0 is a singular point if a function $f(z)$ is not analytic at z_0 but is analytic at some points of each neighborhood of z_0.

Isolated point

A singular point z_0 of $f(z)$ is said to be isolated point if there is a neighbourhood of z_0 which contains no singular points of $f(z)$ except z_0.i.e. $f(z)$ is analytic in some deleted neighborhood, $0<\lvert z-z_0\rvert<\varepsilon$.

e.g. $f(z)=\frac{z^2+1}{(z-1)(z-2)}$ has two isolated point $z=1$ & $z=2$.

Poles

If principal part of Laurent's series has finite number of terms,

$$\text{i.e. } f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\frac{\mathbf{b_1}}{\mathbf{z-z_0}}+\frac{\mathbf{b_2}}{(\mathbf{z-z_0})^2}+.\ldots..+\frac{\mathbf{b_n}}{(\mathbf{z-z_0})^{\mathbf{n}}}$$

then the singularity $z=z_0$ is said to be pole of order n.

If $b_1\neq 0$ and $b_2=b_3=\cdots\ldots=b_n=0$,then

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\frac{b_1}{z-z_0}$$

the singularity $z=z_0$ is said to be pole of order 1 or a simple pole.

Types of Singularities

Removable singularity

If in the Laurent's series expansion, the principal part is zero.

$$\text{i.e. } f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+0$$

then the singularity $z=z_0$ is said to be removable singularity. (i.e. f(z) is not defined at $z=z_0$ but $\lim_{z\to 0}f(z)$ exists.) e.g. $f(z)=\frac{\sin z}{z}$ is undefined at $z=0$ but $\lim_{z\to 0}\frac{\sin z}{z}=1$.

So, $z=0$ is a removable singularity.

Essential singularity

If in the Laurent's series expansion, the principal part contains an infinite number of terms, then the singularity $z = z_0$ is said to be an essential singularity.

e.g. $\mathrm{f(z)} = \sin\frac{1}{z}$ has an essential singularity at $z = 0$, As $\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} + \cdots$

Residue of a function

If $f(z)$ has a pole at the point $z = z_0$ then the coefficient b_1 of the term $(z - z_0)^{-1}$ in the Laurent's series expansion of $f(z)$ at $z = z_0$ is called the residue of $f(z)$ at $z = z_0$.

Residue of $f(z)$ at $z = z_0$ is denoted by $\underset{z=z_0}{\mathrm{Res}}\, f(z)$.

Technique to find Residue

- ✓ If $f(z)$ has a simple pole at $z = z_0$,then $\mathrm{Res}\big(f(z_0)\big) = \lim\limits_{z\to z_0}(z - z_0)f(z)$.
- ✓ If $f(z) = \frac{P(z)}{Q(z)}$ has a simple pole at $z = z_0$,then $\mathrm{Res}\big(f(z_0)\big) = \frac{P(z_0)}{Q'(z_0)}$.
- ✓ If $f(z)$ has a pole of order n at $z = z_0$,then

$$\mathrm{Res}\big(f(z_0)\big) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{(n-1)}}{dz^{(n-1)}}[(z - z_0)^n f(z)]$$

Exercise-3

C	**Que.1**	Discuss the singularity of the point $z = 0$ for the function $(z) = \frac{\sin z}{z}$.	**Jun-13**
H	**Que.2**	Expand $f(z) = \frac{z-\sin z}{z^2}$ at $z = 0$, classify the singular point $z = 0$.	**May-15**
C	**Que.3**	Classify the poles of $f(z) = \frac{1}{z^2-z^6}$.	**Jun-12**
C	**Que.4**	Find the pole of order of the point $z = 0$ for the function $f(z) = \frac{\sin z}{z^4}$	**Dec-13**
C	**Que.5**	Define residue at simple pole and find the sum of residues of the function $f(z) = \frac{\sin z}{z\cos z}$ at its poles inside the circle $\lvert z\rvert = 2$.	**Dec -10**
C	**Que.6**	Find the residue at $z = 0$ of $f(z) = \frac{1-e^z}{z^3}$.	**Jun-12**
H	**Que.7**	Find the residue at $z = 0$ of $f(z) = z\cos\frac{1}{z}$.	**Dec -11**
C	**Que.8**	Show that the singular point of the function $f(z) = \frac{1-\cosh z}{z^3}$ is a pole. Determine the order m of that pole and corresponding residue.	**Dec-12**
H	**Que.9**	Determine residue at poles $\left(\frac{2z+1}{z^2-z-2}\right)$.	**Dec-15**

C	Que.10	Determine the poles of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ and residue at each pole. Evaluate $\int_c f(z)\ dz$, where c is the circle $\lvert z\rvert = 3$.	Dec-15

Cauchy's Residue Theorem

If $f(z)$ is analytic in a closed curve C except at a finite number of singular points within C,then

$$\int_C f(z)\, dz = 2\pi i \text{ (sum of the residue at the singular points)}$$

Exercise-4

C	Que.1	State Cauchy's residue theorem. Evaluate $\int_C \frac{5z-2}{z(z-1)} dz$,where C is the circle $\lvert z\rvert = 2$. **$[10\pi i]$**	Dec -10
H	Que.2	Evaluate $\oint_C \frac{\cos \pi z^2}{(z-1)(z-2)} dz$, where C is the circle $\lvert z\rvert = 3$. **$[4\pi i]$**	Jun-11
C	Que.3	Using residue theorem, Evaluate $\oint_C \frac{e^z+z}{z^3-z} dz$, Where C: $\lvert z\rvert = \frac{\pi}{2}$. **$\left[\pi i\left(e - 2 + \frac{1}{e}\right)\right]$**	May-15
H	Que.4	Using residue theorem, Evaluate $\oint_C \frac{z^2 \sin z}{4z^2-1} dz$, C: $\lvert z\rvert = 2$. **$\left[\frac{\pi i}{4} \sin \frac{1}{2}\right]$**	Jun-10
H	Que.5	Evaluate $\oint_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)^2(z-2)} dz$, where C is the circle $\lvert z\rvert = 3$. **$[4\pi i(\pi + 1)]$**	Dec -11
C	Que.6	Find the value of the integral $\int_C \frac{2z^2+2}{(z-1)(z^2+9)} dz$ taken counterclockwise around the circle C: $\lvert z - 2\rvert = 2$. **$\left[\frac{4}{5}\pi i\right]$**	Dec -12
C	Que.7	Determine the poles of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ and residue at each pole.Hence evaluate $\int_C f(z)\ dz$, where C: $\lvert z\rvert = 3$. **$[2\pi i]$**	Jun-11
H	Que.8	Determine the poles of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ and the residue at each pole. Hence evaluate $\int_c f(z) dz$, where c is the circle $\lvert z\rvert = 2.5$. **$[2\pi i]$**	Jun-14

C	Que.9	Evaluate $\int_C \frac{dz}{(z^2+1)^2}$, where C: $\|z+i\| = 1$. $\left[-\frac{\pi}{2}\right]$	Jun-12
H	Que.10	Define residue at simple pole. Find the residues at each of its poles of $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$ in the finite z −plane. $\left[\mathbf{2\pi i}\left(\frac{2i-14}{25}\right)\right]$	Dec-14
C	Que.11	Evaluate $\oint_C e^{\frac{3}{z}} dz$, where C is $\|z\| = 1$. $[\mathbf{6\pi i}]$	Dec-13
H	Que.12	Use residues to evaluate the integrals of the function $\frac{\exp(-z)}{z^2}$ around the circle $\|z\| = 3$ in the positive sense. $[\mathbf{-2\pi i}]$	Dec-12
C	Que.13	Evaluate $\oint_C \tan z\, dz$, where C is the circle $\|z\| = 2$. $[\mathbf{-4\pi i}]$	Jun-13
C	Que.14	Evaluate $\oint_C \frac{dz}{\sinh 2z}$, Where C: $\|z\| = 2$. $[\mathbf{-\pi i}]$	Dec-12

Rouche's Theorem

If f(z) and g(z) are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C, then $f(z) + g(z)$ and f(z) have the same number of zeros inside C.

Exercise-5

C	Que.1	Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $\|z\| = 1$ and $\|z\| = 2$ using Rouche's theorem.	Dec -11
H	Que.2	Use Rouche's theorem to determine the number of zeros of the polynomial $z^6 - 5z^4 + z^3 - 2z$ inside the circle $\|z\| = 1$.	Dec-12

Integration round the unit circle

An integral of the type $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$, where $F(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$ can be evaluated by taking $z = e^{i\theta}$.

$$\text{Now, } \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left(\frac{z^2 + 1}{z}\right)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{1}{2i}\left(\frac{z^2 - 1}{z}\right)$$

$$\text{Here, } z = e^{i\theta} \Longrightarrow dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{z}$$

Now, the given integral takes the form $\int_c f(z)dz$, where $f(z)$ is a rational function of z and c is the unit circle $|z| = 1$. This complex integral can be evaluated using the residue theorem.

Integration around a small semi-circle

Using Residue theorem,

$$\oint_c F(z)\ dz = \int_{C_R} F(z)\, dz + \int_{-R}^{R} F(x)\, dx \dots\dots \quad (1)$$

Now, By Cauchy’s Residue Theorem

$$\oint_c F(z)\ dz = 2\pi i\ \times \text{sum of residues inside c.}$$

$$\text{As } R \longrightarrow \infty, \int_{-R}^{R} F(x)\, dx \Longrightarrow \int_{-\infty}^{\infty} F(x)\, dx$$

$$\text{Also, } \int_{C_R} F(z)\, dz \longrightarrow 0$$

$$\text{By Eq.}^1, \int_{-\infty}^{\infty} F(x)\, dx = 2\pi i\ \times\ \text{sum of residues of } f(z) \text{inside the c.}$$

Exercise-1

C	Que.1	Using the residue theorem, evaluate $\int_0^{2\pi} \frac{d\theta}{5-3\sin\theta}$. $\left[\frac{\pi}{2}\right]$	Dec -10 Dec-15
H	Que.2	Using the residue theorem, evaluate $\int_0^{2\pi} \frac{4\, d\theta}{5+4\sin\theta}$. $\left[\frac{2\pi}{3}\right]$	Dec -11
T	Que.3	Evaluate $\int_0^{\pi} \frac{d\theta}{17-8\cos\theta}$, by integrating around a unit circle. $\left[\frac{\pi}{15}\right]$	Jun-11
C	Que.4	Using the residue theorem, evaluate $\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$. $\left[\frac{4\pi}{3\sqrt{3}}\right]$	Jun-13

H	Que.5	Evaluate $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$. $[\boldsymbol{\pi}]$	Dec-12
C	Que.6	Using contour Integration show that $\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$.	Dec-15
C	Que.7	Use residues to evaluate $\int_0^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$. $\left[\frac{\pi}{18}\right]$	Jun-11
H	Que.8	Use residues to evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$. $\left[\frac{\pi}{9}\right]$	
C	Que.9	Let $a > b > 0$. Prove that $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$.	Jun-12
T	Que.10	Evaluate $\int_0^{\infty} \frac{x\sin x}{x^2+9}dx$ using residue. $\left[\frac{\pi}{2e^3}\right]$	Dec -13
H	Que.11	Using the theory of residue, evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$. $\left[\frac{2\pi}{e}\right]$	May-15

Definitions

Conformal Mapping

Suppose two curves c_1 and c_2 intersect at point P in Z-plane and the corresponding two curves $c_1{}'$ and $c_2{}'$ at P' in the W-plane.

If the angle of intersection of the curves at P is same as the angle of intersection of the curve P' in both magnitude and sense, then the transformation is said to be Conformal.

Fixed Point (Invariant Point)

Fixed points of mapping $w = f(z)$ are points that are mapped onto themselves are "kept fixed" under the mapping.

Critical Point

The point where $f'(z) = 0$ are called Critical Point.

Ordinary Point

The point where $f'(z) \neq 0$ is called Ordinary Point.

Exercise-1

C	Que 1.	Find Fixed point of bilinear trans. (I) $\mathrm{w} = \dfrac{\mathrm{z}}{2-\mathrm{z}}$ (II) $\mathrm{w} = \dfrac{(2+\mathrm{i})\mathrm{z}-2}{\mathrm{z}+\mathrm{i}}$ (III) $\mathrm{w} = \dfrac{3\mathrm{iz}+1}{\mathrm{Z}+\mathrm{i}}$ $[(\mathbf{I})\boldsymbol{\alpha} = \mathbf{0}, \boldsymbol{\beta} = \mathbf{1}(\mathbf{II})\boldsymbol{\alpha} = \mathbf{1}+\mathbf{i}, \boldsymbol{\beta} = \mathbf{1}-\mathbf{i}\ (\mathbf{III})\boldsymbol{\alpha} = \mathbf{i}]$	
C	Que 2.	Find fixed point of $\mathrm{w} = \frac{\mathrm{z}+1}{\mathrm{z}}$ and verify your result. $\left[\boldsymbol{\alpha} = \frac{\mathbf{1}}{\mathbf{2}} + \frac{\sqrt{\mathbf{5}}}{\mathbf{2}}, \boldsymbol{\beta} = \frac{\mathbf{1}}{\mathbf{2}} - \frac{\sqrt{\mathbf{5}}}{\mathbf{2}}\right]$	Jun-12
C	Que 3.	Define Critical point & find critical point of the $\mathrm{w} = \mathrm{z} + \mathrm{z}^2$. $\left[\mathbf{z} = -\frac{\mathbf{1}}{\mathbf{2}}\right]$	
C	Que 4.	What does conformal mapping mean? At what points is the mapping by $\mathrm{w} = \mathrm{z}^2 + \frac{1}{\mathrm{z}^2}$ not conformal? $[\mathbf{z} = \pm\mathbf{1}, \pm\mathbf{i}]$	Jun -14

Elementary Transformation

Exercise-2

C	Que 1.	Find and sketch the image of the region $\|\mathrm{z}\| > 1$ under the transformation $\mathrm{w} = 4\mathrm{z}$. $[\|\mathbf{w}\| > \mathbf{4}]$	Jun-14
H	Que 2.	Determine & sketch the image of $\|\mathrm{z}\| = 1$ under the transformation $\mathrm{w} = \mathrm{z} + \mathrm{i}$. $[\mathbf{u}^2 + (\mathbf{v}-\mathbf{1})^2 = \mathbf{1}]$	Jun-12
C	Que 3.	Show that the region in the z- plane given by $\mathrm{x} > 0, 0 < \mathrm{y} < 2$ has the image $-1 < \mathrm{u} < 1, \mathrm{v} > 0$ in the w-plane under the transformation $\mathrm{w} = \mathrm{iz} + 1$.	Jun-11

H	Que 4.	Find the image of infinite strip $0 \le x \le 1$ under the transformation $w = iz + 1$. Sketch the region in $\omega -$ plane. $[\mathbf{0 \le v \le 1}]$	
C	Que 5.	Find & sketch (plot) the image of the region $x \ge 1$ under the transformation $w = \frac{1}{z}$. $\left[\left\|\boldsymbol{w} - \frac{1}{2}\right\| \le \frac{1}{2}\right]$	Nov-10
H	Que 6.	Find the image of infinite strip $\frac{1}{4} \le y \le \frac{1}{2}$ under trans. $w = \frac{1}{z}$. [**Region between the circles** $\mathbf{\|w + 2i\| \le 2}$ **&** $\mathbf{\|w + i\| \ge 1}$]	Nov-10
C	Que 7.	Find image of critical $\|z\| = 1$ under transformation $w = f(z) = \frac{z-i}{1-iz}$ & find fixed points. $[\mathbf{v = 0}]$	Nov-11
C	Que 8.	Find the image in the w −plane of the circle $\|z - 3\| = 2$ in the $z -$ plane under the inversion mapping $w = \frac{1}{z}$.	Dec-15
H	Que 9.	Explain translation, rotation and magnification transformation. Find the image of the $\|z - 1\| = 1$ under transformation $w = \frac{1}{z}$. $\left[\mathbf{u = \frac{1}{2}}\right]$	Dec-13
C	Que 10.	Find the image of region bounded by $1 \le r \le 2$ & $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$ in the z- Plane under the transformation $w = z^2$. Show the region graphically. $\left[\mathbf{1 \le r' \le 4} \ \& \ \boldsymbol{\frac{\pi}{3} \le \theta' \le \frac{2\pi}{3}}\right]$	Nov-11
C	Que 11.	Determine the points where $w = z + \frac{1}{z}$ is not conformal mapping. Also find image of circle $\|z\| = 2$ under the transformation $w = z + \frac{1}{z}$. $\left[\mathbf{\frac{u^2}{25} + \frac{v^2}{9} = \frac{1}{4}}\right]$	May-15

Bilinear Transformation / Linear Fractional / Mobius Transformation

A transformation of the form $w = \frac{az+b}{cz+d}$; Where a,b,c,d are complex constants and $ad - bc \ne 0$ is called a Bilinear Transformation.

Determination of Bilinear Transformation

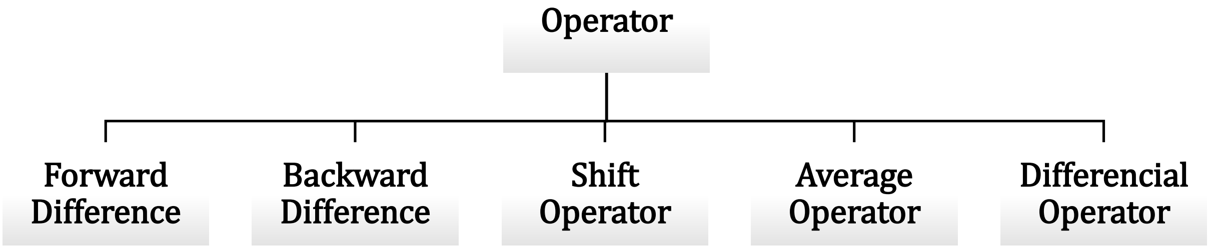
If $w_1, w_2, w_3$are the respective images of distinct points z_1, z_2, z_3 then

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

Exercise-3

C	Que 1.	Determine bilinear transformation which maps point $0, \infty, i$ into $\infty, 1, 0$. $\left[\mathbf{w = \frac{z - i}{z}}\right]$	Jun-12

C	Que 2.	Define Mobius transformation. Determine the Mobius transformation that maps $z_1 = 0, z_2 = 1, z_3 = \infty$ on to $w_1 = -1, w_2 = -i, w_3 = 1$ respectively. $\left[\mathbf{w = -\left(\frac{1 + iz}{1 - iz}\right)}\right]$	Jun-10
H	Que 3.	Determine bilinear transformation which maps point $0, i, 1$ into $i, -1, \infty$. $\left[\mathbf{w = i\left(\frac{i - z}{i + z}\right)}\right]$	May-15
H	Que 4.	Define Mobius transformation. Determine the Mobius transformation that maps $z_1 = 0, z_2 = 1, z_3 = \infty$ onto $w_1 = -1, w_2 = -i, w_3 = 11$ respectively. $\left[\mathbf{w = \left(\frac{z - i}{z + i}\right)}\right]$	Dec-15
H	Que 5.	Determine the Linear Fractional Transformation that maps $z_1 = 0, z_2 = 1, z_3 = \infty$ onto $w_1 = -1, w_2 = -i, w_3 = 1$ respectively. $\left[\mathbf{w = \frac{z - i}{z + i}}\right]$	Jun -14
T	Que 6.	Find bilinear transformation, which maps the points $1, -1, \infty$ onto the points $1 + i, 1 - i, 1$.Also find fixed point. $\left[\mathbf{w = 1 + \frac{i}{z}}\right]$	Dec-12
C	Que 7.	Find the bilinear transformation that maps the points $z_1 = 1, z_2 = i, z_3 = -1$ on to $w_1 = -1, w_2 = 0, w_3 = 1$ respectively. Find image of $\lvert z\rvert < 1$ under this transformation. $\left[\mathbf{w = \frac{z - i}{iz - 1}}\right]$	Dec-13
C	Que 8.	Define a Linear Fractional Transformation. Find the bilinear transformation that maps the points $z_1 = -1, z_2 = 0, z_3 = 1$ on to $w_1 = -i, w_2 = 1, w_3 = i$ respectively. Also find w for $z = \infty$. $\left[\mathbf{w = \frac{i-z}{i+z}}\right]$**(when $z \rightarrow \infty$, $w = -1$)**	Jun-13
T	Que 9.	Find bilinear transformation, which maps the point $z = 1, i, -1$ on to the point $w = i, 0, -i$ .Hence find the image of $\lvert z\rvert < 1$. $\left[\mathbf{w = \frac{1 + iz}{1 - iz}}\right]$	Jun-11
H	Que 10.	Find the Bilinear transformation which maps $z = 1. i. -1$ into $w = 2, i, -2$. $\left[\mathbf{w = \frac{1 + iz}{1 - iz}}\right]$	Dec-15



SR. NO.	TOPIC NAME
1.	Definition of Operators
2.	Relation Between Operators
3.	Newton’s Forward Difference Formula
4.	Newton’s Backward Difference Formula
5.	Gauss’s Forward Difference Formula
6.	Gauss’s Backward Difference Formula
7.	Stirling Formula
8.	Newton’s Divided Difference
9.	Newton’s Divided Difference Formula
10.	Lagrange’s interpolation Formula

Definition of Operators

Forward difference[Δ]	$\Delta f(x) = f(x+h) - f(x)$
Backward difference [∇]	$\nabla f(x) = f(x) - f(x-h)$
Central difference [δ]	$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$
Average Operator [μ]	$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$
Shift Operator [E]	$Ef(x) = f(x+h)$
Differential Operator [D]	$Df(x) = \frac{d}{dx}f(x) = f'(x)$

Relation Between Operators

1. $E = 1 + \Delta$ **(Jun-13, Dec-14)**

Proof

$$(1+\Delta)f(x) = f(x) + \Delta f(x) = f(x) + f(x+h) - f(x) = Ef(x)$$

2. $E\nabla = \Delta$

Proof

$$E\nabla\big(f(x)\big) = E\big(\nabla f(x)\big) = E\big(f(x) - f(x-h)\big) = Ef(x) - Ef(x-h)$$

$$= f(x+h) - f(x)$$

$$= \Delta f(x)$$

$\Rightarrow E\nabla\big(f(x)\big) = \Delta f(x);\ \forall\ f(x)$

$\Rightarrow E\nabla = \Delta$

3. $\Delta\nabla = \Delta - \nabla$

Proof

$$\Delta\nabla\big(f(x)\big) = \Delta\big(\nabla f(x)\big) = \Delta\big(f(x) - f(x-h)\big) = \Delta f(x) - \Delta f(x-h)$$

$$= [f(x+h) - f(x)] - [f(x) - f(x-h)]$$

$$= \Delta f(x) - \nabla f(x) = (\Delta - \nabla)f(x)$$

$\Longrightarrow \Delta\nabla\big(\mathrm{f(x)}\big) = (\Delta - \nabla)\mathrm{f(x)}; \forall\, \mathrm{f(x)}$

$\Longrightarrow \Delta\nabla = \Delta - \nabla$

4. $\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \Delta + \nabla$

Proof

$$\frac{\Delta}{\nabla} - \frac{\nabla}{\Delta} = \frac{\Delta^2 - \nabla^2}{\Delta \cdot \nabla} = \frac{(\Delta - \nabla)\cdot(\Delta + \nabla)}{\Delta - \nabla} = \Delta + \nabla$$

5. $(1 + \Delta)(1 - \nabla) = 1$

Proof

$(1 + \Delta)(1 - \nabla)$

$= 1 - \nabla + \Delta - \Delta.\nabla$

$= 1 - \nabla + \Delta - (\Delta - \nabla) = 1$

6. $\nabla = 1 - \mathrm{E}^{-1}$ **(Dec-13,Dec-14)**

Proof

$$1 - \mathrm{E}^{-1} = 1 - (1 + \Delta)^{-1} = 1 - \frac{1}{1 + \Delta}$$

$$= \frac{1 + \Delta - 1}{1 + \Delta} = \frac{\Delta}{1 + \Delta} = \frac{\mathrm{E}\nabla}{\mathrm{E}} = \nabla$$

7. $\mathrm{E} = \mathrm{e}^{\mathrm{hD}}$ **(Dec-15)**

Proof

$\mathrm{Ef(x)} = \mathrm{f(x + h)} = \mathrm{f(x)} + \mathrm{hf'(X)} + \frac{\mathrm{h}^2}{2!}\mathrm{f''(x)} + \cdots$ (By Taylor's expansion)

$$= \mathrm{f(x)} + \mathrm{hDf(X)} + \frac{\mathrm{h}^2}{2!}\mathrm{D}^2\mathrm{f(x)} + \cdots$$

$$= \left[1 + \mathrm{hD} + \frac{\mathrm{h}^2}{2!}\mathrm{D}^2 + \cdots\right]\mathrm{f(x)}$$

$\Longrightarrow \mathrm{Ef(x)} = \mathrm{e}^{\mathrm{hD}}\mathrm{f(x)} \Longrightarrow \mathrm{E} = \mathrm{e}^{\mathrm{hD}}$

8. $\Delta = \mathrm{e}^{\mathrm{hD}} - 1$ **OR** $\mathrm{hD} = \log(1 + \Delta)$ **(Dec-14, Dec-15)**

Proof

We Know that, $\mathrm{E} = \mathrm{e}^{\mathrm{hD}}$.

Taking $\mathrm{E} = 1 + \Delta \Longrightarrow \Delta = \mathrm{e}^{\mathrm{hD}} - 1 \Longrightarrow \mathrm{hD} = \log(1 + \Delta)$

Newton's Forward Difference Formula

- ✓ If data are $(x_0, y_0), (x_1, y_1)(x_2, y_2), \dots, (x_n, y_n)$
- ✓ $x_0, x_1, x_2, \dots, x_n$ are equally spaced then.

$$f(x) = \mathbf{y} = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots;\ p = \frac{x - x_0}{h}$$

Difference Table

$\mathbf{x}$	$\mathbf{f(x) = y}$	$\mathbf{\Delta f(x)}$	$\mathbf{\Delta^2 f(x)}$	$\mathbf{\Delta^3 f(x)}$	$\mathbf{\Delta^4 f(x)}$
$\mathbf{x_0}$	y_0				
		Δy_0			
$\mathbf{x_1}$	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
$\mathbf{x_2}$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
$\mathbf{x_3}$	y_3		$\Delta^2 y_2$		
		Δy_3			
$\mathbf{x_4}$	y_4				

Exercise-1

<table>
<tr>
<td>C</td>
<td>Que 1.</td>
<td>Construct Newton’s forward interpolation polynomial for the following data.
<table>
<tr><td>X</td><td>4</td><td>6</td><td>8</td><td>10</td></tr>
<tr><td>Y</td><td>1</td><td>3</td><td>8</td><td>16</td></tr>
</table>
Use it to find the value of y for $x = 5$.
$\left[\mathbf{y(x)} = \frac{\mathbf{3x^2-22x+48}}{\mathbf{8}}\ ,\mathbf{y(5)} = \frac{\mathbf{13}}{\mathbf{8}}\right]$</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>Que 2.</td>
<td>Find $\sin 52^0$ using the following values.
<table>
<tr><td>sin 45°</td><td>sin 50°</td><td>sin 55°</td><td>sin 60°</td></tr>
<tr><td>0.7071</td><td>0.7660</td><td>0.8192</td><td>0.8660</td></tr>
</table>
$[\mathbf{0.7880}]$</td>
<td>Nov-11</td>
</tr>
<tr>
<td>H</td>
<td>Que 3.</td>
<td>Use Newton’s forward difference method to find the approximate value of f(2.3) from the following data.
<table>
<tr><td>X</td><td>2</td><td>4</td><td>6</td><td>8</td></tr>
<tr><td>f(x)</td><td>4.2</td><td>8.2</td><td>12.2</td><td>16.2</td></tr>
</table>
$[\mathbf{4.8}]$</td>
<td>Dec-13</td>
</tr>
<tr>
<td>H</td>
<td>Que 4.</td>
<td>Use Newton’s forward difference method to find the approximate value of f(1.3) from the following data.
<table>
<tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>f(x)</td><td>1.1</td><td>4.2</td><td>9.3</td><td>16.4</td></tr>
</table>
$[\mathbf{1.82}]$</td>
<td>Jun-13</td>
</tr>
<tr>
<td>T</td>
<td>Que 5.</td>
<td>Using Newton’s forward formula , find the value off(1.6),if
<table>
<tr><td>x</td><td>1</td><td>1.4</td><td>1.8</td><td>2.2</td></tr>
<tr><td>f(x)</td><td>3.49</td><td>4.82</td><td>5.96</td><td>6.5</td></tr>
</table>
$[\mathbf{5.4394}]$</td>
<td>Jun-11</td>
</tr>
</table>

<table>
<tr>
<td>H</td>
<td>Que 6.</td>
<td>Determine the polynomial by Newton’s forward difference formula from the following table.
<table>
<tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr>
<tr><td>y</td><td>-10</td><td>-8</td><td>-8</td><td>-4</td><td>10</td><td>40</td></tr>
</table>
$[\mathbf{x^3-4x^2+5x-10}]$</td>
<td>Jun-12</td>
</tr>
<tr>
<td>T</td>
<td>Que 7.</td>
<td>Using Newton’s forward interpolation formula ,find the value of f(218),if
<table>
<tr><td>x</td><td>100</td><td>150</td><td>200</td><td>250</td><td>300</td><td>350</td><td>400</td></tr>
<tr><td>f(x)</td><td>10.63</td><td>13.03</td><td>15.04</td><td>16.81</td><td>18.42</td><td>19.90</td><td>21.27</td></tr>
</table>
$[\mathbf{15.47}]$</td>
<td>Jun -14</td>
</tr>
</table>

Newton's Backward Difference Formula

- ✓ If data are $(x_0, y_0), (x_1, y_1)(x_2, y_2), \dots, (x_n, y_n)$.
- ✓ $x_0, x_1, x_2, \dots, x_n$ are equally spaced then.

$$f(x) = \mathbf{y} = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots;\ p = \frac{x - x_n}{h}$$

Difference Table

$\mathbf{x}$	$\mathbf{f(x) = y}$	$\boldsymbol{\nabla}\mathbf{f(x)}$	$\boldsymbol{\nabla}^{\mathbf{2}}\mathbf{f(x)}$	$\boldsymbol{\nabla}^{\mathbf{3}}\mathbf{f(x)}$	$\boldsymbol{\nabla}^{\mathbf{4}}\mathbf{f(x)}$
$\mathbf{x_0}$	y_0				
		∇y_1			
$\mathbf{x_1}$	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
$\mathbf{x_2}$	y_2		$\nabla^2 y_3$		$\boxed{\nabla^4 y_4}$
		∇y_3		$\boxed{\nabla^3 y_4}$	
$\mathbf{x_3}$	y_3		$\boxed{\nabla^2 y_4}$		
		$\boxed{\nabla y_4}$			
$\mathbf{x_4}$	$\boxed{y_4}$				

Exercise-2

<table>
<tr><td>C</td><td>Que 1.</td><td>The area of circle of diameter d is given by
<table>
<tr><td>d</td><td>80</td><td>85</td><td>90</td><td>95</td><td>100</td></tr>
<tr><td>A</td><td>5026</td><td>5674</td><td>6361</td><td>7088</td><td>7854</td></tr>
</table>
Use suitable interpolation to find area of circle of diameter 98. Also Calculate the error.
$[\mathbf{A = 7543.0672, ExactA = 7542.9640, Error = 0.1032}]$</td><td></td></tr>
<tr><td>H</td><td>Que 2.</td><td>Find the cubic polynomial which takes the following values :
$y(0) = 1, y(1) = 0, y(2) = 1,$and $y(3) = 10.$Hence, obtain $y(4)$.
$[\mathbf{y(x) = x^3 - 2x^2 + 1,\ \ y(4) = 33}]$</td><td></td></tr>
<tr><td>H</td><td>Que 3.</td><td>Consider following tabular values
<table>
<tr><td>X</td><td>50</td><td>100</td><td>150</td><td>200</td><td>250</td></tr>
<tr><td>Y</td><td>618</td><td>724</td><td>805</td><td>906</td><td>1032</td></tr>
</table>
Determiney(300).
$[\mathbf{1,148}]$</td><td>Jun-12
Dec-15</td></tr>
<tr><td>T</td><td>Que 4.</td><td>The population of the town is given below. Estimate the population for the year 1925.
<table>
<tr><td>year</td><td>1891</td><td>1901</td><td>1911</td><td>1921</td><td>1931</td></tr>
<tr><td>Population in thousand</td><td>46</td><td>66</td><td>81</td><td>93</td><td>101</td></tr>
</table>
$[\mathbf{96.8368}]$</td><td>May-15</td></tr>
<tr><td>T</td><td>Que 5.</td><td>The following table gives distance (in nautical miles) of the visible horizon for the heights (in feet) above earth’s surface. Find the values of y when x = 390 feet.
<table>
<tr><td>Height(x)</td><td>100</td><td>150</td><td>200</td><td>250</td><td>300</td><td>350</td><td>400</td></tr>
<tr><td>Distance(y)</td><td>10.63</td><td>13.03</td><td>15.04</td><td>16.81</td><td>18.42</td><td>19.90</td><td>21.47</td></tr>
</table>
$[\mathbf{21.004}]$</td><td>Dec-15</td></tr>
</table>

<table>
<tr>
<td>H</td>
<td>Que 6.</td>
<td>From the following table of half-yearly premium for policies maturing at different ages, estimate the premium at the age of 63.
<table>
<tr><td>Age</td><td>45</td><td>50</td><td>55</td><td>60</td><td>65</td></tr>
<tr><td>Premium (in $)</td><td>114.84</td><td>96.16</td><td>83.32</td><td>74.48</td><td>68.48</td></tr>
</table>
$[\mathbf{70.5852}]$</td>
<td>May-15</td>
</tr>
<tr>
<td>H</td>
<td>Que 7.</td>
<td>Compute $f(x) = e^x$ at $x = 0.02$ and $x = 0.38$ Using suitable interpolation formula for the data given below.
<table>
<tr><td>X</td><td>0.0</td><td>0.1</td><td>0.2</td><td>0.3</td><td>0.4</td></tr>
<tr><td>f(x)</td><td>1.0000</td><td>1.1052</td><td>1.2214</td><td>1.3499</td><td>1.4918</td></tr>
</table>
$[\mathbf{1.0202, 1.4623}]$</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>Que 8.</td>
<td>From the following table, find P when $t = 142^0$ c and 175^0 c .using appropriate Newton's interpolation formula.
<table>
<tr><td>Temp. t^0 C</td><td>140</td><td>150</td><td>160</td><td>170</td><td>180</td></tr>
<tr><td>Pressure P</td><td>3685</td><td>4854</td><td>6302</td><td>8076</td><td>10225</td></tr>
</table>
$[\mathbf{P_{142^\circ} = 3898.6688\ ;\ P_{175^\circ} = 9100.4844}]$</td>
<td>Dec-14</td>
</tr>
<tr>
<td>C</td>
<td>Que 9.</td>
<td>The population of the town is given below. Estimate the population for the year 1895 and 1930 using suitable interpolation.
<table>
<tr><td>year</td><td>1891</td><td>1901</td><td>1911</td><td>1921</td><td>1931</td></tr>
<tr><td>Population in thousand</td><td>46</td><td>66</td><td>81</td><td>93</td><td>101</td></tr>
</table>
$\begin{bmatrix}\textbf{Population of year 1895} = \mathbf{54.85} \\ \textbf{population of year 1930} = \mathbf{100.47}\end{bmatrix}$</td>
<td>May-15</td>
</tr>
<tr>
<td>H</td>
<td>Que 10.</td>
<td>Compute values of f(0.12) and f(0.40) using suitable interpolation formula for the following data:
<table>
<tr><td>x</td><td>0.10</td><td>0.15</td><td>0.20</td><td>0.25</td><td>0.30</td></tr>
<tr><td>f(x)</td><td>0.1003</td><td>0.1511</td><td>0.2027</td><td>0.2553</td><td>0.3093</td></tr>
</table>
$[\mathbf{f(0.12) = 0.1205, f(0.40) = 0.4241}]$</td>
<td>Dec-15</td>
</tr>
</table>

<table>
<tr>
<td>T</td>
<td>Que 11.</td>
<td>Compute cosh(0.56) & cosh(0.76) from the following table.
<table>
<tr><td>x</td><td>0.5</td><td>0.6</td><td>0.7</td><td>0.8</td></tr>
<tr><td>cosh x</td><td>1.127626</td><td>1.185465</td><td>1.255169</td><td>1.337435</td></tr>
</table>
$[\mathbf{1.16095}, \mathbf{1.30297}]$</td>
<td>Nov-10
Jun-10</td>
</tr>
<tr>
<td>C</td>
<td>Que 12.</td>
<td>Using Newton’s suitable formula , find the value of f(1.6) & f(2),if
<table>
<tr><td>x</td><td>1</td><td>1.4</td><td>1.8</td><td>2.2</td></tr>
<tr><td>f(x)</td><td>3.49</td><td>4.82</td><td>5.96</td><td>6.5</td></tr>
</table>
$[\mathbf{5.4394}, \mathbf{6.3306}]$</td>
<td>Jun-11</td>
</tr>
</table>

Gauss’s Forward Difference Formula

- ✓ If data are $(x_{-n}, y_{-n}), \dots, (x_{-2}, y_{-2}), (x_{-1}, y_{-1}), (x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.
- ✓ $x_0, x_1, x_2, \dots, x_n$ are equally spaced then.

$$f(x) = y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \cdots;\ p = \frac{x - x_0}{h}$$

Difference Table

$\boldsymbol{x}$	$\boldsymbol{f(x) = y}$	$\Delta \boldsymbol{f(x)}$	$\Delta^2 \boldsymbol{f(x)}$	$\Delta^3 \boldsymbol{f(x)}$	$\Delta^4 \boldsymbol{f(x)}$
$\boldsymbol{x_{-2}}$	y_{-2}				
		Δy_{-2}			
$\boldsymbol{x_{-1}}$	y_{-1}		$\Delta^2 y_{-2}$		
		Δy_{-1}		$\Delta^3 y_{-2}$	
$\boldsymbol{x_0}$	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$	
$\boldsymbol{x_1}$	y_1		$\Delta^2 y_0$		
		Δy_1			
$\boldsymbol{x_2}$	y_2				

Gauss's Backward Difference Formula

- ✓ If data are $(x_{-n}, y_{-n}), \dots, (x_{-2}, y_{-2}), (x_{-1}, y_{-1}), (x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.
- ✓ $x_0, x_1, x_2, \dots, x_n$ are equally spaced then.

$$f(x) = y = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \cdots;\ p = \frac{x - x_0}{h}$$

Difference Table

$\boldsymbol{x}$	$\boldsymbol{f(x) = y}$	$\Delta \boldsymbol{f(x)}$	$\Delta^2 \boldsymbol{f(x)}$	$\Delta^3 \boldsymbol{f(x)}$	$\Delta^4 \boldsymbol{f(x)}$
$\boldsymbol{x_{-2}}$	y_{-2}				
		Δy_{-2}			
$\boldsymbol{x_{-1}}$	y_{-1}		$\Delta^2 y_{-2}$		
		Δy_{-1}		$\Delta^3 y_{-2}$	
$\boldsymbol{x_0}$	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$	
$\boldsymbol{x_1}$	y_1		$\Delta^2 y_0$		
		Δy_1			
$\boldsymbol{x_2}$	y_2				

Stirling Formula

- ✓ If data are $(x_{-n}, y_{-n}), \dots, (x_{-2}, y_{-2}), (x_{-1}, y_{-1}), (x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$.
- ✓ $x_0, x_1, x_2, \dots, x_n$ are equally spaced then.

$$f(x) = y = y_0 + p\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right]$$
$$+ \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\left[\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right] + \cdots$$

$$\text{Where, } p = \frac{x - x_0}{h}$$

Difference Table

$\mathbf{x}$	$\mathbf{f(x) = y}$	$\Delta\mathbf{f(x)}$	$\Delta^2\mathbf{f(x)}$	$\Delta^3\mathbf{f(x)}$	$\Delta^4\mathbf{f(x)}$
$\mathbf{x_{-2}}$	y_{-2}				
		Δy_{-2}			
$\mathbf{x_{-1}}$	y_{-1}		$\Delta^2 y_{-2}$		
		Δy_{-1}		$\Delta^3 y_{-2}$	
$\mathbf{x_0}$	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$	
$\mathbf{x_1}$	y_1		$\Delta^2 y_0$		
		Δy_1			
$\mathbf{x_2}$	y_2				

Exercise-3

<table>
<tr>
<td>H</td>
<td>Que 1.</td>
<td>Apply Stirling’s formula to compute y(35) from the following table.
<table>
<tr><td>X</td><td>20</td><td>30</td><td>40</td><td>50</td></tr>
<tr><td>Y</td><td>512</td><td>439</td><td>346</td><td>243</td></tr>
</table>
$[\mathbf{394.69}]$</td>
<td>Jun-11</td>
</tr>
<tr>
<td>H</td>
<td>Que 2.</td>
<td>Let $f(40) = 836,\ f(50) = 682,\ f(60) = 436,\ f(70) = 272$ use Stirling’s Method to findf(55).
$[\mathbf{565.0625}]$</td>
<td>Jun-12</td>
</tr>
<tr>
<td>C</td>
<td>Que 3.</td>
<td>Using Stirling’s formula find y_{35} by using given data.
<table>
<tr><td>X</td><td>10</td><td>20</td><td>30</td><td>40</td><td>50</td></tr>
<tr><td>Y</td><td>600</td><td>512</td><td>439</td><td>346</td><td>243</td></tr>
</table>
$[\mathbf{395.430}]$</td>
<td></td>
</tr>
</table>

Newton’s Divided Difference

- ✓ $[x_0, x_1] = \Delta\!\!\!|\, y_0 \;=\; \frac{f(x_1)-f(x_0)}{x_1-x_0}$
- ✓ $[x_0, x_1, x_2] = \Delta\!\!\!|^{\,2} y_0 \;=\; \frac{[x_1,x_2]-[x_0,x_1]}{x_2-x_0}$
- ✓ $[x_0, x_1, \dots, x_n] = \Delta\!\!\!|^{\,n} y_0 = \frac{[x_1,x_2,\dots,x_n]-[x_0,x_1,\dots,x_{n-1}]}{x_n-x_0}$

Newton’s Divided Difference Formula

$$f(x) = y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \cdots$$

Divided Difference Table

$\mathbf{x}$	$\mathbf{f(x) = y}$	$\Delta\!\!\!|\,\mathbf{f(x)}$	$\Delta\!\!\!|^{\,2}\mathbf{f(x)}$	$\Delta\!\!\!|^{\,3}\mathbf{f(x)}$	$\Delta\!\!\!|^{\,4}\mathbf{f(x)}$
$\mathbf{x_0}$	y_0				
		$\Delta\!\!\!|\, y_0 = \frac{y_1 - y_0}{x_1 - x_0}$			
$\mathbf{x_1}$	y_1		$\Delta\!\!\!|^{\,2} y_0 = \frac{\Delta\!\!\!|\, y_1 - \Delta\!\!\!|\, y_0}{x_2 - x_0}$		
		$\Delta\!\!\!|\, y_1 = \frac{y_2 - y_1}{x_2 - x_1}$		$\Delta\!\!\!|^{\,3} y_0 = \frac{\Delta\!\!\!|^{\,2} y_1 - \Delta\!\!\!|^{\,2} y_0}{x_3 - x_0}$	
$\mathbf{x_2}$	y_2		$\Delta\!\!\!|^{\,2} y_1 = \frac{\Delta\!\!\!|\, y_2 - \Delta\!\!\!|\, y_1}{x_3 - x_1}$		$\Delta\!\!\!|^{\,4} y_0 = \frac{\Delta\!\!\!|^{\,3} y_1 - \Delta\!\!\!|^{\,3} y_0}{x_4 - x_0}$
		$\Delta\!\!\!|\, y_2 = \frac{y_3 - y_2}{x_3 - x_2}$		$\Delta\!\!\!|^{\,3} y_1 = \frac{\Delta\!\!\!|^{\,2} y_2 - \Delta\!\!\!|^{\,2} y_1}{x_4 - x_1}$	
$\mathbf{x_3}$	y_3		$\Delta\!\!\!|^{\,2} y_2 = \frac{\Delta\!\!\!|\, y_3 - \Delta\!\!\!|\, y_2}{x_4 - x_2}$		
		$\Delta\!\!\!|\, y_3 = \frac{y_4 - y_3}{x_4 - x_3}$			
$\mathbf{x_4}$	y_4				

Exercise-4

<table>
<tr><td>C</td><td>Que 1.</td><td>Define Divided diff- interpolation formula.</td><td>Nov-10</td></tr>
<tr><td>C</td><td>Que 2.</td><td>If $f(x) = \frac{1}{x}$,find the divided differences $[a, b]$ and $[a, b, c]$.
$\left[-\frac{\mathbf{1}}{\boldsymbol{ab}} \ \boldsymbol{and} \ \frac{\mathbf{1}}{\boldsymbol{abc}}\right]$</td><td>Jun-10</td></tr>
<tr><td>H</td><td>Que 3.</td><td>From the following table, find Newton’s divided difference formula
<table>
<tr><td>x</td><td>1</td><td>2</td><td>7</td><td>8</td></tr>
<tr><td>$f(x)$</td><td>1</td><td>5</td><td>5</td><td>4</td></tr>
</table>
$\left[\boldsymbol{f(x) = 1 + 4(x - 1) - \frac{2}{3}(x^2 - 3x + 2) + \frac{1}{14}(x^3 - 10x^2 + 23x - 14}\right]$</td><td>Jun-11</td></tr>
<tr><td>C</td><td>Que 4.</td><td>The Shear stress in kips, per square foot for 5 specimens in a clay stratum are:
<table>
<tr><td>Depth(m)</td><td>1.9</td><td>3.1</td><td>4.2</td><td>5.1</td><td>5.8</td></tr>
<tr><td>Stress(ksf)</td><td>0.3</td><td>0.6</td><td>0.4</td><td>0.9</td><td>0.7</td></tr>
</table>
Use Newton’s divided difference formula to compute the stress at 4.5m depth.
$[\mathbf{0.54}]$</td><td>Dec-12</td></tr>
<tr><td>T</td><td>Que 5.</td><td>Write a formula for divided difference $[x_0, x_1]$ & $[x_0, x_1, x_2]$ using Newton’s divided difference formula compute $f(9.5)$ from the following data
<table>
<tr><td>x</td><td>8</td><td>9</td><td>9.2</td><td>11</td></tr>
<tr><td>$f(x)$</td><td>2.079442</td><td>2.197225</td><td>2.219203</td><td>2.397895</td></tr>
</table>
$[\mathbf{2.251284}]$</td><td>Dec-13</td></tr>
<tr><td>T</td><td>Que 6.</td><td>Write a formula for divided difference $[x_0, x_1]$ & $[x_0, x_1, x_2]$ using Newton’s divided difference formula compute $f(10.5)$ from the following data
<table>
<tr><td>x</td><td>10</td><td>11</td><td>13</td><td>17</td></tr>
<tr><td>$f(x)$</td><td>2.3026</td><td>2.3979</td><td>2.5649</td><td>2.8332</td></tr>
</table>
$[\mathbf{2.3596}]$</td><td>Jun-13</td></tr>
</table>

<table>
<tr>
<td>C</td>
<td>Que 7.</td>
<td>Compute $f(8)$from the following value using Newton's Divided difference formula.
<table>
<tr><td>x</td><td>4</td><td>5</td><td>7</td><td>10</td><td>11</td><td>13</td></tr>
<tr><td>$f(x)$</td><td>48</td><td>100</td><td>294</td><td>900</td><td>1210</td><td>2028</td></tr>
</table>
$[\mathbf{448}]$</td>
<td>Nov-11
Dec-11</td>
</tr>
<tr>
<td>H</td>
<td>Que 8.</td>
<td>Compute $f(9.2)$ from the following value using Newton's divided difference.
<table>
<tr><td>x</td><td>8</td><td>9</td><td>9.5</td><td>11.0</td></tr>
<tr><td>$f(x)$</td><td>2.079442</td><td>2.197225</td><td>2.251292</td><td>2.397895</td></tr>
</table>
$[\mathbf{2.219297}]$</td>
<td>Jun-10
Nov-10
Dec-15</td>
</tr>
<tr>
<td>H</td>
<td>Que 9.</td>
<td>Given following data for the polynomial $f(x) = 3x^3 - 5x^2 + 4x + 1$. Compute $f(0.3)$ using Newton's divided difference formula.
<table>
<tr><td>x</td><td>0</td><td>1</td><td>3</td><td>4</td><td>7</td></tr>
<tr><td>$f(x)$</td><td>1</td><td>3</td><td>49</td><td>129</td><td>813</td></tr>
</table>
$[\mathbf{1.831}]$</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>Que 10.</td>
<td>Evaluate $f(9)$, using Newton's divided difference from the following data
<table>
<tr><td>x</td><td>5</td><td>7</td><td>11</td><td>13</td><td>17</td></tr>
<tr><td>$f(x)$</td><td>150</td><td>392</td><td>1452</td><td>2366</td><td>5202</td></tr>
</table>
$[\mathbf{810}]$</td>
<td>Jun-14</td>
</tr>
<tr>
<td>T</td>
<td>Que 11.</td>
<td>Construct Divided difference table for the data given below
<table>
<tr><td>x</td><td>-4</td><td>-1</td><td>0</td><td>2</td><td>5</td></tr>
<tr><td>f(x)</td><td>1245</td><td>33</td><td>5</td><td>9</td><td>1335</td></tr>
</table>
$[\Delta^4 \mathbf{f(x) = 3 \text{ or } [x_0, x_1, x_2, x_3, x_4] = 3}]$</td>
<td>May-15</td>
</tr>
</table>

H	Que 12.	Using Newton's divided difference formula find f(3) from the following table. x: -1, 2, 4, 5 F(x): -5, 13, 255, 625 $[\mathbf{53.68}]$	Dec-15

x	-1	2	4	5
F(x)	-5	13	255	625

Lagrange's Interpolation Formula

- ✓ If data are $(x_0, y_0), (x_1, y_1)(x_2, y_2), \dots, (x_n, y_n)$.
- ✓ $x_0, x_1, x_2, \dots, x_n$ are unequally spaced then.

$$y = f(x) = \frac{(x - x_1)(x - x_2) \dots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_n)} y_0 + \frac{(x - x_0)(x - x_2) \dots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \dots (x_1 - x_n)} y_1 + \cdots + \frac{(x - x_0)(x - x_1) \dots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \dots (x_n - x_{n-1})} y_n$$

Lagrange's Inverse Interpolation Formula

- ✓ If data are $(x_0, y_0), (x_1, y_1)(x_2, y_2), \dots, (x_n, y_n)$.
- ✓ $x_0, x_1, x_2, \dots, x_n$ are unequally spaced then.

$$x = \frac{(y - y_1)(y - y_2) \dots (y - y_n)}{(y_0 - y_1)(y_0 - y_2) \dots (y_0 - y_n)} x_0 + \frac{(y - y_0)(y - y_2) \dots (y - y_n)}{(y_1 - y_0)(y_1 - y_2) \dots (y_1 - y_n)} x_1 + \cdots + \frac{(y - y_0)(y - y_1) \dots (y - y_{n-1})}{(y_n - y_0)(y_n - y_1) \dots (y_n - y_{n-1})} x_n$$

Exercise-5

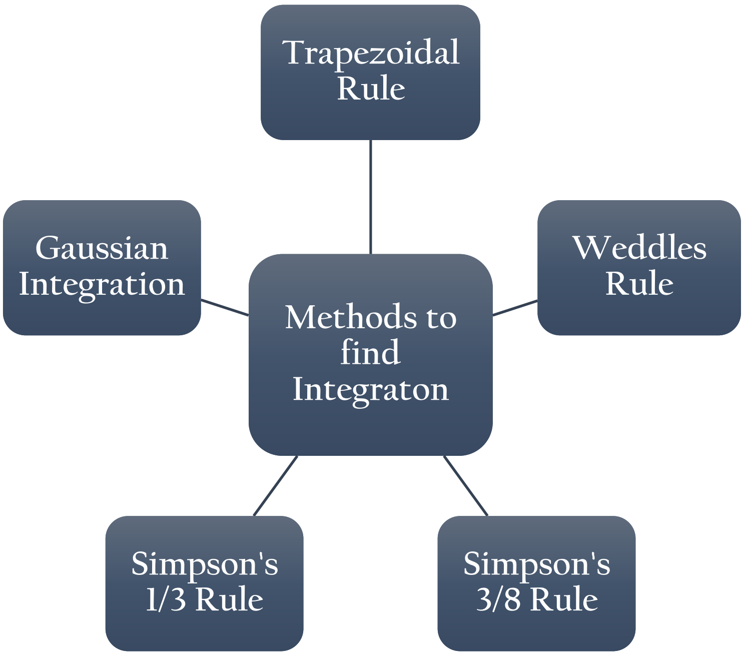
C	Que 1.	Determine the interpolating polynomial of degree three using Lagrange's interpolation for the table below. $\left[\mathbf{P(x) = \frac{1}{24}(x^3 - 25x + 24)}\right]$	Jun-10

x	−1	0	1	3
y	2	1	0	−1

<table>
<tr>
<td>H</td>
<td>Que 2.</td>
<td>Explain quadratic Lagrange’s interpolation .Compute f(2) by using Lagrange’s interpolation method from the following data
<table>
<tr><td>x</td><td>-1</td><td>0</td><td>1</td><td>3</td></tr>
<tr><td>f(x)</td><td>2</td><td>1</td><td>0</td><td>-1</td></tr>
</table>
$[\mathbf{-0.75}]$</td>
<td>Jun-13</td>
</tr>
<tr>
<td>H</td>
<td>Que 3.</td>
<td>Find the Lagrange’s interpolating polynomial from the following data. Also, Compute f(4) .
<table>
<tr><td>x</td><td>2</td><td>3</td><td>5</td><td>7</td></tr>
<tr><td>f(x)</td><td>0.1506</td><td>0.3001</td><td>0.4517</td><td>0.6259</td></tr>
</table>
$[\mathbf{0.3896}]$</td>
<td>Dec-12</td>
</tr>
<tr>
<td>C</td>
<td>Que 4.</td>
<td>Explain quadratic Lagrange’s interpolation .Compute f(9.2) by using Lagrange’s interpolation method from the following data
<table>
<tr><td>x</td><td>9</td><td>9.5</td><td>11</td></tr>
<tr><td>f(x)</td><td>2.1972</td><td>2.2513</td><td>2.3979</td></tr>
</table>
$[\mathbf{2.2192}]$</td>
<td>Dec-13</td>
</tr>
<tr>
<td>T</td>
<td>Que 5.</td>
<td>Using Lagrange’s formula of fit a polynomial to the data.
<table>
<tr><td>x</td><td>-1</td><td>0</td><td>2</td><td>3</td></tr>
<tr><td>y</td><td>8</td><td>3</td><td>1</td><td>12</td></tr>
</table>
And hence find y(2).
$$\mathbf{y(x) = \frac{1}{3}[2x^3 + 2x^2 - 15x + 9], y(2) = 1}$$</td>
<td>DEC-11</td>
</tr>
<tr>
<td>T</td>
<td>Que 6.</td>
<td>Find the Lagrange’s interpolating polynomial from the following data.
<table>
<tr><td>x</td><td>0</td><td>1</td><td>4</td><td>5</td></tr>
<tr><td>y</td><td>1</td><td>3</td><td>24</td><td>39</td></tr>
</table>
$$\left[\mathbf{P(x) = \frac{1}{20}[3x^3 + 10x^2 + 27x + 20]}\right]$$</td>
<td>Nov-10</td>
</tr>
</table>

<table>
<tr>
<td>C</td>
<td>Que 7.</td>
<td>From the following data find value of x when $y = f(x) = 0.39$.
<table>
<tr><td>x</td><td>20</td><td>25</td><td>30</td></tr>
<tr><td>Y=f(x)</td><td>0.342</td><td>0.423</td><td>0.500</td></tr>
</table>
$[\mathbf{x = 22.841}]$</td>
<td>Dec-15</td>
</tr>
<tr>
<td>C</td>
<td>Que 8.</td>
<td>Find the Lagrange’s interpolating polynomial from the following data. Find x,when $f(x) = 13.00$.
<table>
<tr><td>x</td><td>44</td><td>45</td><td>46</td><td>47</td></tr>
<tr><td>f(x)</td><td>13.40</td><td>13.16</td><td>12.93</td><td>12.68</td></tr>
</table>
$[\mathbf{45.69}]$</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>Que 9.</td>
<td>Apply Lagrange’s formula to find a root of the equation$f(x) = 0$.
<table>
<tr><td>x</td><td>30</td><td>34</td><td>38</td><td>42</td></tr>
<tr><td>f(x)</td><td>−30</td><td>−13</td><td>3</td><td>18</td></tr>
</table>
$[\mathbf{37.23}]$</td>
<td></td>
</tr>
<tr>
<td>H</td>
<td>Que 10.</td>
<td>Find the Lagrange’s interpolating polynomial from the following data. And Evaluate f(9)
<table>
<tr><td>x</td><td>5</td><td>7</td><td>11</td><td>13</td><td>17</td></tr>
<tr><td>f(x)</td><td>150</td><td>392</td><td>1452</td><td>2366</td><td>5202</td></tr>
</table>
$[\mathbf{810}]$</td>
<td>Jun -14</td>
</tr>
<tr>
<td>T</td>
<td>Que 11.</td>
<td>Find y(12) by Lagrange’s Interpolation formula from following values.
<table>
<tr><td>x</td><td>11</td><td>13</td><td>14</td><td>18</td><td>20</td><td>23</td></tr>
<tr><td>y</td><td>25</td><td>47</td><td>68</td><td>82</td><td>102</td><td>124</td></tr>
</table>
$[\mathbf{y(12) = 26.41}]$</td>
<td>Dec-14</td>
</tr>
</table>

<table>
<tr>
<td>C</td>
<td>Que 12.</td>
<td>By Lagrange's interpolation formula, Obtain the value of $f(x) = 85$.
<table>
<tr><td>x</td><td>2</td><td>5</td><td>8</td><td>14</td></tr>
<tr><td>y</td><td>94.8</td><td>97.9</td><td>81.3</td><td>68.7</td></tr>
</table>
$[\mathbf{4.6141}]$</td>
<td>Dec-15</td>
</tr>
<tr>
<td>H</td>
<td>Que 13.</td>
<td>Find the Lagrange's interpolation polynomial from the following data. Also find f(2).
<table>
<tr><td>x</td><td>0</td><td>1</td><td>4</td><td>5</td></tr>
<tr><td>f(x)</td><td>1</td><td>3</td><td>24</td><td>39</td></tr>
</table>
$[\mathbf{6.9}]$</td>
<td>Dec-15</td>
</tr>
<tr>
<td>T</td>
<td>Que 14.</td>
<td>Determine the interpolating polynomial of degree three by using Lagrange's interpolation for the following data. Also find f(2).
<table>
<tr><td>X</td><td>-1</td><td>0</td><td>1</td><td>3</td></tr>
<tr><td>F(x)</td><td>2</td><td>1</td><td>0</td><td>-1</td></tr>
</table>
$[\mathbf{f(2) = -0.75}]$</td>
<td>Dec-15</td>
</tr>
<tr>
<td>C</td>
<td>Que 15.</td>
<td>Express the function $\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}$ as a sum of partial fractions, using Lagrange's Formula.
$$\left[\frac{\mathbf{1}}{\boldsymbol{x-1}}+\frac{\mathbf{1}}{\boldsymbol{x-2}}+\frac{\mathbf{1}}{\boldsymbol{x-3}}\right]$$</td>
<td>Jun-11</td>
</tr>
<tr>
<td>H</td>
<td>Que 16.</td>
<td>Express the function $\frac{2x^2+3x+5}{(x-1)(x+2)(x-2)}$ as a sum of partial fractions, using Lagrange's formula.
$$\left[\frac{\mathbf{-10}}{\boldsymbol{3(x-1)}}+\frac{\mathbf{7}}{\boldsymbol{12(x+2)}}+\frac{\mathbf{19}}{\boldsymbol{4(x-2)}}\right]$$</td>
<td></td>
</tr>
</table>



SR. NO.	TOPIC NAME
1	Newton-Cotes Formula
1	Trapezoidal Rule
2	Simpson’s 1/3 – Rule
3	Simpson’s 3/8 – Rule
4	Weddle’s Rule
5	Gaussian Integration(Gaussian Quadrature)

Newton-Cotes Formula

$$\int_a^b f(x)dx = h\left[ny_0 + n^2\,\Delta y_0 + \left(\frac{n^3}{6} - \frac{n^2}{4}\right)\Delta^2 y_0 + \left(\frac{n^4}{24} + \frac{n^3}{6} - \frac{n^2}{4}\right)\Delta^3 y_0 + \cdots\right]$$

$$\text{Where, } h = \frac{b-a}{n}$$

Trapezoidal Rule (If n is a multiple of 1.)

$$\int_a^b f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]\,;\, h = \frac{b-a}{n}$$

Exercise-1

T	**Que.1**	State Trapezoidal rule with $n = 10$ and evaluate $\int_0^1 e^{-x^2} dx$. $[\mathbf{0.7462}]$	**Nov-10**
C	**Que.2**	Write Trapezoidal rule for numerical integration.	**Jun-13**
C	**Que.3**	State Trapezoidal rule with $n = 10$ and evaluate $\int_0^1 e^x\, dx$. $[\mathbf{1.7197}]$	**Jun-11**
H	**Que.4**	State Trapezoidal rule with $n = 10$ and using it, evaluate $\int_0^1 2e^x\, dx$. $[\mathbf{3.4394}]$	**Dec-14**
T	**Que.5**	Evaluate $\int_0^1 \frac{dx}{1+x^2}$using trapezoidal rule with $h = 0.2$. $[\mathbf{0.7837}]$	**Nov-11** **Jun-11**
H	**Que.6**	Evaluate $\int_0^1 \frac{dx}{1+x^2}$using trapezoidal rule. $[\mathbf{0.7854}]$	**May-15**
C	**Que.7**	Given the data below, find the isothermal work done on the gas as it is compressed from $v_1 = 22L$ to $v_2 = 2L$ use $w = -\int_{v_1}^{v_2} P\;dv$ V,L: 2, 7, 12, 17, 22 P(atm.): 12.20, 3.49, 2.04, 1.44, 1.11 Use Trapezoidal rule. $[\mathbf{68.125}]$	**Dec-12**
H	**Que.8**	Consider the following tabular values. x: 25, 25.1, 25.2, 25.3, 25.4, 25.5, 25.6 $y = f(x)$: 3.205, 3.217, 3.232, 3.245, 3.256, 3.268, 3.28 Determine the area bounded by the given curve and X-axis between $x = 25$ to $x = 25.6$ by Trapezoidal rule. $[\mathbf{1.9461}]$	**Jun -12**
H	**Que.9**	Evaluate $\int_0^6 \frac{dx}{1+x^2}$ dx by using Trapezoidal rule taking $h = 1$. $[\mathbf{1.4108}]$	**Jun -14**

C	Que.10	Evaluate $\int_0^{\pi} \sin x \, dx$, taking $n = 10$. $[\mathbf{1.7182}]$	Dec-15
H	Que.11	Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule taking $h = 0.2$ $[\mathbf{0.7837}]$	Dec-15
T	Que.12	Evaluate $\int_1^5 \log_{10} x \; dx$ taking 8 subintervals by Trapezoidal rule. $[\mathbf{1.7505}]$	Dec-15

Simpson's $1/3$- Rule (If n is a multiple of 2.)

$$\int_a^b f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

$$; h = \frac{b-a}{n}$$

Exercise-2

<table>
<tr><td>C</td><td>Que.1</td><td>Evaluate $\int_0^6 \frac{dx}{1+x}$ taking $h = 1$using Simpson's $\frac{1}{3}$ rule.Hence Obtain an approximate value of $\log_e 7$.
$[\mathbf{1.9588}]$</td><td>Jun-11</td></tr>
<tr><td>T</td><td>Que.2</td><td>Derive Trapezoidal rule and Evaluate $\int_{0.5}^{1.3} e^{x^2} dx$ by using simpson's $\frac{1}{3}^{rd}$ rule.
$[\mathbf{2.0762}]$</td><td>Dec-15</td></tr>
<tr><td>H</td><td>Que.3</td><td>Using Simpson's $\frac{1}{3}$ rule evaluate $\int_1^{2.5} f(x)dx$ from the following data. Take $h = 0.3$.
<table>
<tr><td>X</td><td>1</td><td>1.3</td><td>1.6</td><td>1.9</td><td>2.2</td><td>2.5</td></tr>
<tr><td>f(x)</td><td>1</td><td>1.69</td><td>2.56</td><td>3.61</td><td>4.84</td><td>6.25</td></tr>
</table>
$[\mathbf{4.325}]$</td><td>Jun-13</td></tr>
<tr><td>H</td><td>Que.4</td><td>Using Simpson's$^1/_3$ rule evaluate $\int_0^6 f(x)dx$ from following data. $h = 1$.
<table>
<tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr>
<tr><td>f(x)</td><td>1</td><td>0.5</td><td>0.3333</td><td>0.25</td><td>0.2</td><td>0.1666</td><td>0.1428</td></tr>
</table>
$[\mathbf{1.9586}]$</td><td>Dec-13</td></tr>
</table>

<table>
<tr>
<td>T</td>
<td>Que.5</td>
<td>The speed , v meter per second , of a car , t seconds after it starts, is shown in the following table
<table>
<tr><td>t</td><td>0</td><td>12</td><td>24</td><td>36</td><td>48</td><td>60</td><td>72</td><td>84</td><td>96</td><td>108</td><td>120</td></tr>
<tr><td>v</td><td>0</td><td>3.6</td><td>10.08</td><td>18.90</td><td>21.6</td><td>18.54</td><td>10.26</td><td>4.5</td><td>4.5</td><td>5.4</td><td>9</td></tr>
</table>
Using Simpson’s 1/3 rule, find the distance travelled by the car in 2minutes.
$[\mathbf{1222.56}]$</td>
<td>Jun-10
Dec-10</td>
</tr>
<tr>
<td>C</td>
<td>Que.6</td>
<td>A river is 80 meter wide. The depth ‘d’ in meters at a distance x meters from one bank is given by the following table, calculate the area of cross-section of the river using Simpson’s $\frac{1}{3}$ rule
<table>
<tr><td>x</td><td>0</td><td>10</td><td>20</td><td>30</td><td>40</td><td>50</td><td>60</td><td>70</td><td>80</td></tr>
<tr><td>d</td><td>0</td><td>4</td><td>7</td><td>9</td><td>12</td><td>15</td><td>14</td><td>8</td><td>3</td></tr>
</table>
$[\mathbf{710}]$</td>
<td>Dec-11
May-15</td>
</tr>
<tr>
<td>T</td>
<td>Que.7</td>
<td>Consider the following tabular values.Find $\int_{10}^{16} y dx$ by Simpson $\frac{1}{3}$ rule.
<table>
<tr><td>x</td><td>10</td><td>11</td><td>12</td><td>13</td><td>14</td><td>15</td><td>16</td></tr>
<tr><td>y</td><td>1.02</td><td>0.94</td><td>0.89</td><td>0.79</td><td>0.71</td><td>0.62</td><td>0.55</td></tr>
</table>
$[\mathbf{4.7233}]$</td>
<td>Jun-12</td>
</tr>
<tr>
<td>C</td>
<td>Que.8</td>
<td>Evaluate $\int_{-2}^{6}(1+x^2)^{\frac{3}{2}} dx$ using Simpson’s $\frac{1}{3}$ rule with taking 6 subintervals. Use four digits after decimal point for calculation.
$[\mathbf{360.1830}]$</td>
<td>Dec-12</td>
</tr>
<tr>
<td>H</td>
<td>Que.9</td>
<td>Evaluate $\int_{0}^{6} \frac{1}{1+x^2} dx$ by using Simpson’s $\frac{1}{3}$ rule taking $h = 1$.
$[\mathbf{1.3662}]$</td>
<td>Jun-14</td>
</tr>
<tr>
<td>C</td>
<td>Que.10</td>
<td>A solid of revolution is formed rotating about x-axis, the lines $x = 0$, $x = 1$ and a curve through the points with the following coordinates.
<table>
<tr><td>X</td><td>0</td><td>0.25</td><td>0.5</td><td>0.75</td><td>1</td></tr>
<tr><td>Y</td><td>1</td><td>0.9896</td><td>0.9598</td><td>0.9089</td><td>0.8415</td></tr>
</table>
Estimate the volume of the solid formed using Simpson’s rule.
$[\mathbf{1.1059}]$</td>
<td>May-15</td>
</tr>
<tr>
<td>H</td>
<td>Que.11</td>
<td>Evaluate $\int_{0}^{\pi} \sin x \, dx$, Take $n = 10$
$[\mathbf{1.7182}]$</td>
<td>Dec-15</td>
</tr>
</table>

Simpson’s 3/8- Rule (If n is a multiple of 3.)

$$\int_a^b f(x)dx = \frac{3h}{8}[(y_0 + y_n) + 2(y_3 + y_6 + \cdots + y_{n-3}) + 3(y_1 + y_2 + y_4 + y_5 + \cdots + y_{n-2} + y_{n-1})]$$

$$; h = \frac{b-a}{n}$$

Exercise-3

C	Que.1	Write the Simpson’s $\frac{3}{8}$ rule for numerical integration.	Dec-13
T	Que.2	Evaluate $\int_0^3 \frac{dx}{1+x}$ with $n = 6$ by using Simpson’s $\frac{3}{8}$ rule and hence calculate $\log_e 2$.Estimate the bound of error involved in the process. $[\mathbf{1.3888, 0.0563}]$	Jun-10 Jun-14
H	Que.3	State Simpson’s $\frac{3}{8}$ rule and evaluate $\int_0^1 \frac{1}{1+x^2} dx$ taking $h = \frac{1}{6}$.and also by Simpson’s $\frac{1}{3}$ rule taking $h = \frac{1}{4}$. $[\mathbf{0.7854}]$	Dec-10 May-15 Dec-15
C	Que.4	Evaluate the integral $\int_4^{5.2} \log_e x\, dx$ using Simpson’s $\frac{3}{8}$ rule. $[\mathbf{1.8278}]$	Dec-11
C	Que.5	Dividing the range into 10 equal part, evaluate $\int_0^{\pi} \sin x\; dx$ by simpson’s $\frac{3}{8}$ rule. $[\mathbf{1.99}]$	May-15
T	Que.6	Evaluate $\int_0^1 \frac{dx}{1+x}$ using Simpson’s $\frac{3}{8}$ rule. $[\mathbf{0.6937}]$	Dec-15

Weddle's Rule (If n is multiple of 6.)

$$\int_a^b f(x)\,dx = \frac{3h}{10}\left[\begin{array}{c}(y_0+y_n)+(5y_1+y_2+6y_3+y_4+5y_5)+(2y_6+5y_7+y_8+6y_9+y_{10}+5y_{11})\\ +\cdots+(2y_{n-6}+5y_{n-5}+y_{n-4}+6y_{n-3}+y_{n-2}+5y_{n-1})\end{array}\right]$$

$$;h = \frac{b-a}{n}$$

Exercise-4

C	Que.1	Consider the following tabular values. x: 25, 25.1, 25.2, 25.3, 25.4, 25.5, 25.6 f(x): 3.205, 3.217, 3.232, 3.245, 3.256, 3.268, 3.28 Determine the area bounded by the given curve and X-axis between $x = 25$ to $x = 25\ .6$ by Weddle's rule. $[\mathbf{1.9460}]$	Jun -12
H	Que.2	Consider the following tabular values. Find $\int_{10}^{16} ydx$ by Weddle's rule. x: 10, 11, 12, 13, 14, 15, 16 y: 1.02, 0.94, 0.89, 0.79, 0.71, 0.62, 0.55 $[\mathbf{4.713}]$	Jun -12

Que.1 data:

x	25	25.1	25.2	25.3	25.4	25.5	25.6
f(x)	3.205	3.217	3.232	3.245	3.256	3.268	3.28

Que.2 data:

x	10	11	12	13	14	15	16
y	1.02	0.94	0.89	0.79	0.71	0.62	0.55

Gaussian integration (Gaussian Quadrature)

$$\int_a^b f(x)dx = \frac{b-a}{2}\big(w_1 f(y_1) + w_2\ f(y_2) + \cdots + w_n\ f(y_n)\big), \text{Where } x = \frac{b-a}{2}y + \frac{b+a}{2}$$

Table

n	w_i	y_i
1	2.0000	0.0000
2	1.0000	−0.57735
	1.0000	0.57735
3	0.55555	−0.77460
	0.88889	0.00000
	0.55555	0.77460
4	0.34785	−0.86114
	0.65214	−0.33998
	0.65214	0.33998
	0.34785	0.86114
5	0.23693	−0.90618
	0.47863	−0.53847
	0.56889	0.00000
	0.47863	0.53847
	0.23693	0.90618

You can also use following formula to find Gaussian Quadrature.

- ✓ One Point Gaussian Quadrature Formula $(n = 1)$

$$\int_{-1}^{1} f(x)dx = 2f(0)$$

- ✓ Two Point Gaussian Quadrature Formula $(n = 2)$

$$\int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

- ✓ Three Point Gaussian Quadrature Formula $(n = 3)$

$$\int_{-1}^{1} f(x)dx = \frac{8}{9}f(0) + \frac{5}{9}\left(f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right)$$

Exercise-5

C	Que.1	Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ by using Gaussian quadrature formula with one point, two points & three points. $[\mathbf{2}, \mathbf{1.5}, \mathbf{1.58333}]$	Dec-15
C	Que.2	Evaluate $I = \int_{0}^{1} \frac{dt}{1+t}$ by Gaussian formula with one point, two-point and three- points. $[\mathbf{0.66667}, \mathbf{0.69231}, \mathbf{0.69312}]$	Dec-11 Dec-15
T	Que.3	Evaluate $\int_{0}^{1} e^{-x^2}\,dx$ by Gauss integration formula with $n = 3$. $[\mathbf{0.74681}]$	Jun-10 Dec-14 May-15
T	Que.4	Evaluate $\int_{1}^{3} sinx\ \ dx$ using Gauss Quadrature of five points. Compare the result with analytic value. $[\mathbf{1.53031}, \mathbf{1.53029}]$	Nov-10
H	Que.5	Evaluate integral $\int_{-2}^{6} (1+x^2)^{3/2}\,dx$ by the Gaussian formula for $n = 3$. $[\mathbf{358.69236}]$	Dec-12

Matrix Equation

The matrix notation for following linear system of equation is as follow:

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{array}\right\}$$

$$\text{Here } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & & a_{3n} \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \; B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

The above linear system is expressed in the matrix form as $A \cdot X = B$.

Elementary Transformation or Operation on a Matrix

	Operation	Meaning
✓	R_{ij} or $R_i \leftrightarrow R_j$	Interchange of i^{th} and j^{th} rows
✓	$k \cdot R_i$	Multiplication of all the elements of i^{th} row by non zero scalar k.
✓	$R_{ij}(k)$ or $R_j + k \cdot R_i$	Multiplication of all the elements of i^{th} row by nonzero scalar k and added into j^{th} row.

Row Echelon Form of Matrix

To convert the matrix into row echelon form follow the following steps:

1. Every zero row of the matrix occurs below the non zero rows.
2. Arrange all the rows in strictly decreasing order.
3. Make all the entries zero below the leading (first non zero entry of the row) element of 1st row.
4. Repeat step-3 for each row.

Reduced Row Echelon Form of Matrix:

To convert the matrix into reduced row echelon form follow the following steps:

1. Convert given matrix into row echelon form.
2. Make all leading elements 1(one).
3. Make all the entries zero above the leading element 1(one) of each row.

Gauss Elimination Method

To solve the given linear system using Gauss elimination method, follow the following steps:

1. Start with augmented matrix $[A : B]$.
2. Convert matrix A into row echelon form with leading element of each row is one(1).
3. Apply back substitution for getting equations.
4. Solve the equations and find the unknown variables (i.e. solution).

Exercise-1

Solve the following system of equations by Gauss-elimination method.			
H	**Que. 1**	$x + y + z = 9, 2x - 3y + 4z = 13, 3x + 4y + 5z = 40.$ $[\mathbf{1, 3, 5}]$	**Jun-11**
C	**Que. 2**	$2x + y + z = 10, 3x + 2y + 3z = 18, x + 4y + 9z = 16.$ $[\mathbf{7, -9, 5}]$	
T	**Que. 3**	$x + 2y + z = 3, 2x + 3y + 3z = 10, 3x - y + 2z = 13.$ $[\mathbf{2, -1, 3}]$	
H	**Que. 4**	$2x + 3y - z = 5, 4x + 4y - 3z = 3, 2x - 3y + 2z = 2.$ $[\mathbf{1, 2, 3}]$	
T	**Que. 5**	$2x + y - z = 1, 5x + 2y + 2z = -4, 3x + y + z = 5.$ $[\mathbf{14, -32, -5}]$	**Dec-12**
C	**Que. 6**	$8y + 2z = -7, 3x + 5y + 2z = 8, 6x + 2y + 8z = 26.$ $[\mathbf{4, -1, \frac{1}{2}}]$	**Jun-14**
H	**Que. 7**	$x + 4y - z = -5, x + y - 6z = -12, 3x - y - z = 4.$ $[\mathbf{2.0845, -1.1408, 1.6477}]$	**May-15**
H	**Que. 8**	$x + y + 2z = 4, 3x + y - 3z = -4, 2x - 3y - 5z = -5$ $[\mathbf{1, -1, 2}]$	**May-15**

Exercise-2

Solve the following system of equations using partial pivoting by Gauss-elimination method. (With Partial Pivoting)			
C	**Que. 1**	$8x_2 + 2x_3 = -7, 3x_1 + 5x_2 + 2x_3 = 8, 6x_1 + 2x_2 + 8x_3 = 26.$ $\left[\mathbf{4, -1, \frac{1}{2}}\right]$	**Dec-10** **Jun-10**
T	**Que. 2**	$x + y + z = 7, 3x + 3y + 4z = 24, 2x + y + 3z = 16.$ $[\mathbf{3, 1, 3}]$	**Nov-11** **Dec-15**
H	**Que. 3**	$2x_1 + 2x_2 - 2x_3 = 8, -4x_1 - 2x_2 + 2x_3 = -14$ $-2x_1 + 3x_2 + 9x_3 = 9.$ $[\mathbf{3, 2, 1}]$	
T	**Que. 4**	$2x_1 + 2x_2 + x_3 = 6, 4x_1 + 2x_2 + 3x_3 = 4, x_1 + x_2 + x_3 = 0$ $[\mathbf{5, 1, -6}]$	**Jun-15** **Dec-15**

Gauss Seidel Method

This is a modification of Gauss-Jacobi method. In this method we replace the approximation by the corresponding new ones as soon as they are calculated.

Consider the system of equations.

$$\mathbf{a_1x + b_1y + c_1z = d_1}$$

$$\mathbf{a_2x + b_2y + c_2z = d_2}$$

$$\mathbf{a_3x + b_3y + c_3z = d_3}$$

Where co-efficient matrix **A** must be diagonally dominant,

$$|\mathbf{a_1}| \geq |\mathbf{b_1}| + |\mathbf{c_1}|$$

$$|\mathbf{b_2}| \geq |\mathbf{a_2}| + |\mathbf{c_2}|$$

$$|\mathbf{c_3}| \geq |\mathbf{a_3}| + |\mathbf{b_3}| \dots\dots (\mathbf{1})$$

And the inequality is strictly greater than for at least one row.

Solving the system $(\mathbf{1})$ for $\mathbf{x}, \mathbf{y}, \mathbf{z}$ respectively,we obtain

$$\mathbf{x = \frac{1}{a_1}(d_1 - b_1y - c_1z)}$$

$$\mathbf{y = \frac{1}{b_2}(d_2 - a_2x - c_2z)}$$

$$\mathbf{z = \frac{1}{c_3}(d_3 - a_3x - b_3y)} \dots\dots \ (\mathbf{2})$$

We start with $\mathbf{x_0 = 0, y_0 = 0}$ & $\mathbf{z_0 = 0}$ in equ.$(\mathbf{2})$

$$\therefore \mathbf{x_1 = \frac{1}{a_1}(d_1 - b_1y_0 - c_1z_0)}$$

Now substituting $\mathbf{x = x_1}$ & $\mathbf{z = z_0}$ in the second equ. Of $(\mathbf{2})$

$$\therefore \mathbf{y_1 = \frac{1}{b_2}(d_2 - a_2x_1 - c_2z_0)}$$

Now substituting $\mathbf{x = x_1}$ & $\mathbf{y = y_1}$ in the third equ. Of $(\mathbf{2})$

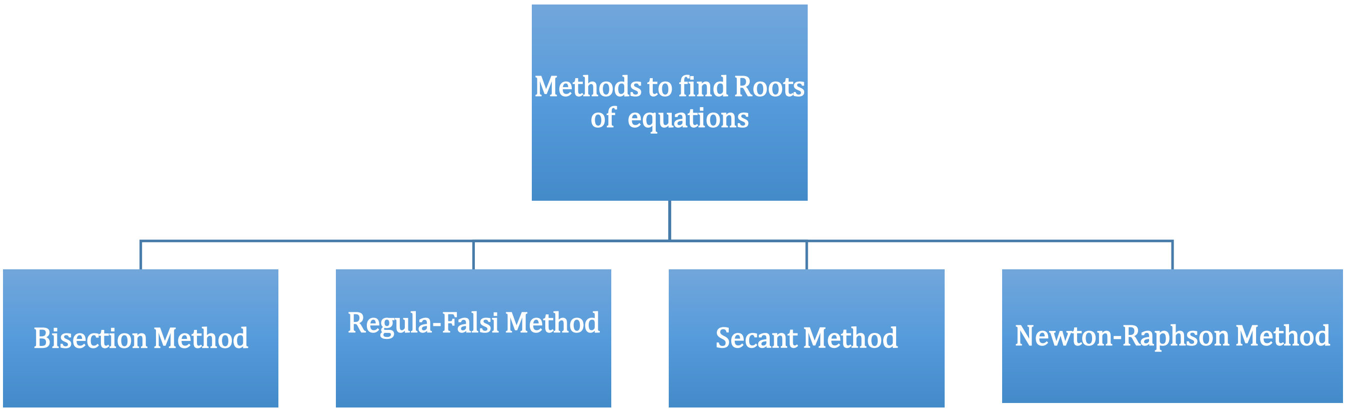
$$\therefore \mathbf{z_1 = \frac{1}{c_3}(d_3 - a_3x_1 - b_3y_1)}$$

This process is continued till the values of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are obtained to desired degree of accuracy.

Exercise-4

C	Que. 1	Solve the following system of equations by Gauss-Seidel method. $10x_1 + x_2 + x_3 = 6$, $x_1 + 10x_2 + x_3 = 6$, $x_1 + x_2 + 10x_3 = 6$. $[\mathbf{0.5}, \mathbf{0.5}, \mathbf{0.5}]$	Jun-10 Dec-15
T	Que. 2	Use Gauss seidel method to determine roots of the following equations. $2x - y = 3$, $x + 2y + z = 3$, $-x + z = 3$. $[\mathbf{1}, \mathbf{-1}, \mathbf{4}]$	Jun-13
H	Que. 3	Use Gauss seidel method to find roots of the following equations. $8x + y + z = 5$, $x + 8y + z = 5$, $x + y + 8z = 5$. $[\mathbf{0.5}, \mathbf{0.5}, \mathbf{0.5}]$	Dec-13 May-15
H	Que. 4	Solve the following system of equations by Gauss-Seidel method. $10x_1 + x_2 + x_3 = 12$, $2x_1 + 10x_2 + x_3 = 13$, $2x_1 + 2x_2 + 10x_3 = 14$. $[\mathbf{1}, \mathbf{1}, \mathbf{1}]$	Dec-10 Dec-15
C	Que. 5	Solve by Gauss-Seidel & Gauss-Jacobi method correct up to two decimal places. $20x + 2y + z = 30, x - 40y + 3z = -75,2x - y + 10z = 30$. $[\mathbf{1.14}, \mathbf{2.13}, \mathbf{2.99}]$	Jun-11
H	Que. 6	Solve this system of linear equations using Jacobi's method in three iterations first check the co-efficient matrix of the following systems is diagonally dominant or not? $20x + y - 2z = 17, 2x - 3y + 20z = 25, 3x + 20y - z = -18$ $[\mathbf{1}, \mathbf{-1}, \mathbf{1}]$	Dec-15
H	Que. 7	Solve the following system of equations by Gauss-Seidel method. $20x + y - 2z = 17, 2x - 3y + 20z = 25, 3x + 20y - z = -18$ $[\mathbf{1}, \mathbf{-1}, \mathbf{1}]$	Jun-14
C	Que. 8	Solve by Gauss-Seidel method correct up to three decimal places. $2x + y + 54z = 110,27\ x + 6y - z = 85,6x + 15y + 2z = 72$. $[\mathbf{2.422}, \mathbf{3.580}, \mathbf{1.881}]$	Nov-11
H	Que. 9	Solve by Gauss-Seidel Method. $9x + 2y + 4z = 20, x + 10y + 4z = 6,2x - 4y + 10z = -15$. $[\mathbf{2.74}, \mathbf{0.99}, \mathbf{-1.65}]$	
H	Que. 10	Solve by Gauss-Seidel method correct up to three decimal places. $10x - 5y - 2z = 3,4x - 10y + 3z = -3, x + 6y + 10z = -3$. $[\mathbf{0.342}, \mathbf{0.285}, \mathbf{-0.505}]$	
T	Que. 11	Check whether the following system is diagonally dominant or not. If not, re-arrange the equations so that it becomes diagonally dominant and hence solve the system of simultaneous linear equation by Gauss sidle Method. $-100y + 130z = 230, -40x + 150y - 100z = 0,60x - 40y = 200$. $[\mathbf{7.78}, \mathbf{6.67}, \mathbf{6.90}]$	Dec-12
H	Que. 12	Solve the following system of equations using Gauss-Seidel method correct up to three decimal places. $60x - 4y + 6z = 150$; $2x + 2y + 18z = 30$; $x + 17y - 2z = 48$ $[\mathbf{2.580}, \mathbf{2.798}, \mathbf{1.069}]$	Dec-13
T	Que. 13	State diagonal dominant property .Using Gauss-seidel method solve $6x + y + z = 105$; $4x + 8y + 3z = 155$; $5x + 4y - 10z = 65$ $[\mathbf{15}, \mathbf{10}, \mathbf{5}]$	May-15

T	Que. 14	By gauss Seidel method solve the following system upto six iteration $12x_1 + 3x_2 - 5x_3 = 1$; $x_1 + 5x_2 + 3x_3 = 28$; $3x_1 + 7x_2 + 13x_3 = 76$ Use initial condition $(x_1\ x_2\ x_3) = (1\ 0\ 1)$. $[\mathbf{1}, \mathbf{3}, \mathbf{4}]$	May-15
H	Que. 15	By gauss Seidel method solve the following system $2x + y + 6z = 9$; $8x + 3y + 2z = 13$; $x + 5y + z = 7$ $[\mathbf{1}, \mathbf{1}, \mathbf{1}]$	May-15
H	Que. 16	State the Direct and iterative methods to solve system of linear equations. Using Gauss-Seidel method ,solve $2x_1 - x_2 = 7$; $-x_1 + 2x_2 - x_3 = 1$; $-x_2 + 2x_3 = 1$ $[\mathbf{5.3125}, \mathbf{4.3125}, \mathbf{2.6563}]$	Dec-15



SR.NO.	TOPIC NAME
1	Bisection Method
2	Secant Method
3	Regula-Falsi Method (False Position Method)
5	Newton-Raphson Method

Bisection Method

- ✓ $\mathrm{f(x) = 0}$
- ✓ If $\mathrm{f(a) > 0}$ and $\mathrm{f(b) < 0}$,Where a and b are consecutive integer, then

$$x_1 = \frac{a + b}{2}$$

- ✓ Check $f(x_1) > 0$ OR $f(x_1) < 0$.
- ✓ If $f(x_1) > 0$,then $x_2 = \frac{x_1+b}{2}$

  OR

  If $f(x_1) < 0$,then we find $x_2 = \frac{a+x_1}{2}$.
- ✓ Check $f(x_2) > 0$ OR $f(x_2) < 0$.
- ✓ If $f(x_1) > 0, f(x_2) > 0$ then $x_3 = \frac{x_2+b}{2}$ OR $f(x_1) < 0, f(x_2) > 0$ then $x_3 = \frac{x_2+x_1}{2}$

  OR

  If $f(x_1) > 0, f(x_2) < 0$ then $x_3 = \frac{x_1+x_2}{2}$ OR $f(x_1) < 0, f(x_2) < 0$ then $x_3 = \frac{a+x_2}{2}$

Processing like this when latest two consecutive values of x are not same.

Exercise-1

C	Que.1	Find the positive root of $x = \cos x$ correct up to three decimal places by bisection method. $[\mathbf{0.739}]$	Jun-10
H	Que.2	Solve $x = \cos x$ by Bisection method correct up to two decimal places. $[\mathbf{0.75}]$	Jun-14
C	Que.3	Explain bisection method for solution of equation. Using this method find the approximate solution $x^3 + x - 1 = 0$ of correct up to three decimal points. $[\mathbf{0.683}]$	Dec-13
T	Que.4	Perform the five iterations of the bisection method to obtain a root of the equation $f(x) = \cos x - xe^x = 0$. $[\mathbf{0.53125}]$	Nov-10
T	Que.5	Find root of equation $x^3 - 4x - 9 = 0$, using the bisection method in four stages. $[\mathbf{2.6875}]$	Jun-11

H	Que.6	Perform the five iteration of the bisection method to obtain a root of the equation $x^3 - x - 1 = 0$. $[\mathbf{1.34375}]$	Nov-11
C	Que.7	Find the negative root of $x^3 - 7x + 3 = 0$ bisection method up to three decimal place. $[\mathbf{-2.839}]$	Jun-12
H	Que.8	Find a real root of the following equation by bisection method a) $e^x - 2\cos x = 0$ b) $x^3 - 9x + 1 = 0$ c) $x\log_{10^x} - 1.2 = 0$(up to four stage) $[\mathbf{0.6931, 0.1113, 2.6875}]$	
T	Que.9	Use bisection method to find a root of equation $x^3 + 4x^2 - 10 = 0$ in the interval [1,2].Use four iteration. $[\mathbf{1.3125}]$	Dec-12
H	Que.10	Perform three iterations of Bisection method to obtain root of the equation $2\sin x - x = 0$. $[\mathbf{1.875}]$	May-15
T	Que.11	Explain bisection method for solving an equation $f(x) = 0$.Find the real root of equation $x^2 - 4x - 10 = 0$ by using this method correct to three decimal places. $[\mathbf{5.742}]$	Dec-15

Secant Method

- ✓ $f(x) = 0$
- ✓ $x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n); n = 1,2,3, \dots$

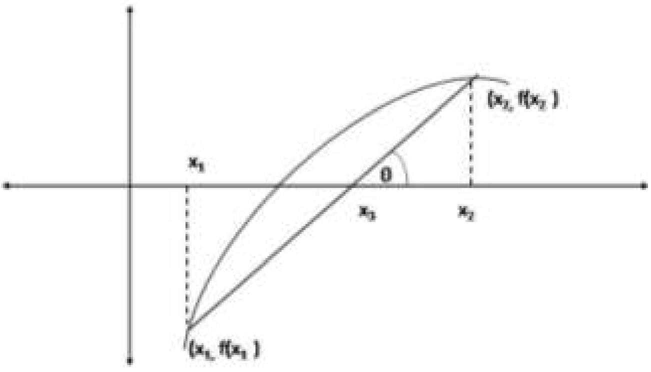
Processing like this when latest two consecutive values of x are not same.

Explanation:

Approximate the graph of $y = f(x)$ by Secant Line determined by two initial points $[x_0, f(x_0)]$ and $[x_1, f(x_1)]$, as shown in figure.

Define x_2 to be the point of intersection of the line(secant) through these two points; then figure shoes that x_2 will be the closer x than either x_0 or x_1. Using the slope formula with secant line, we have

$$m = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \dots (1)$$



$$m = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{f(x_1) - 0}{x_1 - x_2} \quad \dots (2)$$

By Eq. (1) & (2),

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_1) - 0}{x_1 - x_2}$$

$$\Rightarrow x_1 - x_2 = \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

Using x_1 and x_2, repeat this process to obtain x_3 etc.

The general term is given by

$$x_{n+1} = x_n - \left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right) f(x_n); n = 1,2,3, \dots$$

Regula-Falsi Method (False Position Method)

- ✓ $f(x) = 0$.
- ✓ If $f(x_0) \cdot f(x_1) < 0$,Where x_0 and $x_1$are consecutive integer, then we find

$$x_2 = \frac{x_1 f(x_0) - x_0 f(x_1)}{f(x_0) - f(x_1)}$$

- ✓ Check $f(x_2) < 0$ or $f(x_2) > 0$.
- ✓ If $f(x_2) \cdot f(x_1) < 0$,then we find

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

OR

- ✓ If $f(x_2) \cdot f(x_0) < 0$,then we find

$$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)}$$

Processing like this when latest two consecutive values of x are not same.

Exercise-2

C	Que.1	Find the positive solution of $f(x) = x - 2\sin x = 0$ by the secant method, starting form $x_0 = 2, x_1 = 1.9$. $[\mathbf{1.8955}]$	**Nov-10** **Jun-14**
T	Que.2	Derive Secant method and solve $xe^x - 1 = 0$ correct up to three decimal places between 0 and 1. $[\mathbf{0.567}]$	**Jun-12**

H	Que.3	Find the real root of the following by secant method. a) $x^2 - 4x - 10 = 0$ (using $x_0 = 4, x_1 = 2$,upto six iteration) b) $x^3 - 2x - 5 = 0$ (using $x_0 = 2, x_1 = 3$, upto four iteration) $[\mathbf{5.7411, 2.0928}]$	
H	Que.4	Use Secant method to find the roots of $\cos x - xe^x = 0$ correct upto 3 decimal places of decimal. $[\mathbf{0.518}]$	May-15
T	Que.5	Find smallest positive root of an equation $x - e^{-x} = 0$ using Regula Falsi method correct to four significant digits. $[\mathbf{0.6065}]$	May-15
C	Que.6	Apply False Position method to find the negative root of the equation $x^3 - 2x + 5 = 0$ correct to four decimal places. $[\mathbf{-2.0946}]$	May-15
C	Que.7	Find a root of the equation $x^3 - 4x - 9 = 0$ using False-position method correct up to three decimal. $[\mathbf{2.7065}]$	Dec-15
H	Que.8	Explain False position method for finding the root of the equation $f(x) = 0$.Use this method to find the root of an equation $x = e^{-x}$ correct up to three decimal places. $[\mathbf{0.567}]$	Dec-15
H	Que.9	Using method of False-position, compute the real root of the equation $x \log x - 1.2 = 0$ correct to four decimals. $[\mathbf{2.74021}]$	Dec-15

Newton-Raphson Method (Newton's Method)

- ✓ $f(x) = 0$
- ✓ $f(a) \cdot f(b) < 0$
- ✓ $x_0 = a$ when $|f(a)| < |f(b)|$ OR $x_0 = b$ when $|f(b)| < |f(a)|$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; n = 0,1,2,3 \ldots$$

$$\text{Where} \quad f'(x_n) \neq 0$$

Processing like this when latest two consecutive values of x are not same.

Explanation:

Le $x_1$be the root of $f(x) = 0$ and x_0 be an approximation to x_1. If $h = x_1 - x_0$, then by Taylor's Series,

$$f(x_0 + h) = f(x_0) + h\, f'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots$$

Since, $x_1 = x_0 + h$ is root. $f(x_1) = f(x_0 + h) = 0$

If h is chosen too small enough, then we can neglect 2nd , 3rd and higher powers of h.

We have,

$$0 = f(x_0) + h\,f'(x_0) \Rightarrow h = -\frac{f(x_0)}{f'(x_0)}\ ;\ f'(x_0) \neq 0.$$

Suppose that, $x_1 = x_0 + h$ be the better approximation.

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

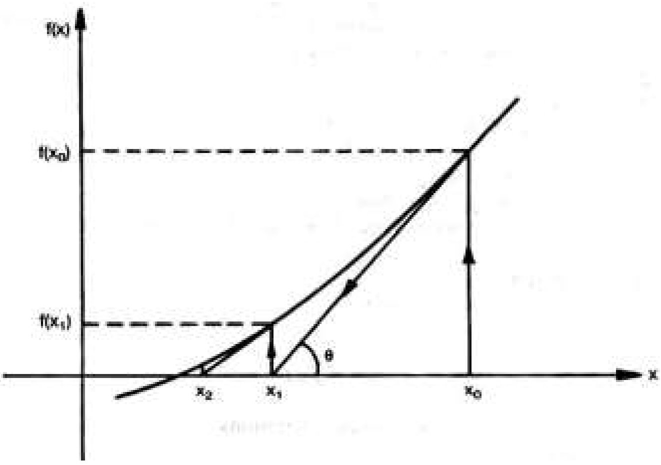
By repeating the process,

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}; n = 0,1,2,3 \ldots$$



Where $f'(x_n) \neq 0$

This is called Newton-Raphson Formula.

Note

If the function is linear then N-R method has to be failed.

Find the iterative formula for $\sqrt{N}$ and $\frac{1}{N}$ by N-R method.

Formula for $\sqrt{N}$

$x = \sqrt{N} \Rightarrow x^2 - N = 0 \Rightarrow f(x) = x^2 - N$

$\Rightarrow f'(x) = 2x$

By N-R formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{{x_n}^2 - N}{2x_n} = \frac{2{x_n}^2 - {x_n}^2 + N}{2x_n}$$

$$= \frac{{x_n}^2 + N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$$

Formula for $\frac{1}{N}$

$$x = \frac{1}{N} \Rightarrow \frac{1}{x} - N = -0 \Rightarrow f(x) = \frac{1}{x} - N$$

$$\Rightarrow f'(x) = -\frac{1}{x^2}$$

By N-R formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}} = x_n + \left(\frac{1}{x_n} - N\right) x_n^2 = 2x_n - N x_n^2$$

$$x_{n+1} = x_n(2 - N x_n)$$

Exercise-3

C	Que.1	Derive the Newton Raphson iterative scheme by drawing appropriate figure.	Nov-10 May-15
C	Que.2	Explain Newton's method for solving equation $f(x) = 0$. Apply this method to find the approximate solution of $x^3 + x - 1 = 0$ correct up to three decimal. $[\mathbf{0.682}]$	Jun-13
H	Que.3	Using Newton-Raphson method, find a root of the equation $x^3 + x - 1 = 0$ correct to four decimal places. $[\mathbf{0.6823}]$	Jun-14
T	Que.4	Find the positive root of $x = \cos x$ correct up to three decimal places by N-R method. $[\mathbf{0.739}]$	Jun-11
T	Que.5	Find to four decimal places, the smallest root of the equation $\sin x = e^{-x}$ Using the N-R starting with $x_0 = 0.6$. $[\mathbf{0.5885}]$	Dec-11
C	Que.6	Obtain Newton-Raphson formula from Taylor's theorem.	Jun-12
C	Que.7	Find a root of $x^4 - x^3 + 10x + 7 = 0$ correct up to three decimal places between $a = -2$ & $b = -1$ by N-R method. $[\mathbf{-1.454}]$	Jun-12
T	Que.8	Find a zero of function $f(x) = x^3 - \cos x$ with starting point $x_0 = 1$ by NR Method could $x_0 = 0$ be used for this problem ? $[\mathbf{0.8655}]$	Dec-12
T	Que.9	Discuss the rate of convergence of NR Method.	Jun-12
C	Que.10	Find an iterative formula to find $\sqrt{N}$ (N is a positive number) and hence find $\sqrt{5}$. $[\mathbf{2.2361}]$	Jun-10
H	Que.11	Explain Newton's method for solving equation $f(x) = 0$. Apply this method to Find an iterative formula to find $\sqrt{N}$ and hence find $\sqrt{7}$ Correct up to three decimal points. $[\mathbf{2.646}]$	Dec-13
T	Que.12	Set up a Newton iteration for computing the square root x of a given positive number c and apply it to $c = 2$. $[\mathbf{1.4142}]$	Nov -10

H	**Que.13**	Find an iterative formula to find $\sqrt{N}$.(Where N is a positive number). Hence find $\sqrt{27}$. $[\mathbf{5.1962}]$	
H	**Que.14**	Find a real root by N-R method. a) $x^3 - 3x - 5 = 0$ b) $x = e^{-x}$ (up to three decimal) c) $3x - \cos x - 1 = 0$ (up to three decimal) $[\mathbf{2.2790, 0.567, 0.607}]$	
C	**Que.15**	Find an iterative formula to find $\frac{1}{N}$(Where N is positive number) and hence evaluat-e $\frac{1}{3}, \frac{1}{19}, \frac{1}{23}$. $[\mathbf{0.333, 0.0526, 0.0435}]$	
H	**Que.16**	Derive an iterative formula to find $\sqrt{N}$ hence find approximate value of $\sqrt{65}$ and $\sqrt{3}$, correct up to three decimal places. $[\mathbf{8.062, 1.732}]$	**Dec-14**
C	**Que.17**	Derive an iterative formula for finding cube root of any positive number using Newton Raphson method and hence find approximate value of $\sqrt[3]{58}$. $[\mathbf{3.8708}]$	**May-15**
H	**Que.18**	Using Newton-Raphson method find a root of the equation $xe^x = 2$ Correct to three decimal places. $[\mathbf{0.518}]$	**Dec-15**
H	**Que.19**	Find the $\sqrt{10}$ correct to three decimal places by using Newton-Raphson iterative method. $[\mathbf{3.1623}]$	**Dec-15**
T	**Que.20**	Use Newton-Raphson method to find smallest positive root of $f(x) = x^3 - 5x + 1 = 0$ correct to four decimals. $[\mathbf{0.20164}]$	**Dec-15**

Power Method

- ✓ Given A $=$ 2×2 matrix or 3×3 matrix.
- ✓ We take initial vector x_0 then find second vector x_1 by $A\,x_0 = \lambda\,x_1$.
- ✓ Find x_2 by $A\,x_1 = \lambda\,x_2$.
- ✓ Processing like this when latest two consecutive values of X are not same.
- ✓ Second Eigen value = Trace of matrix A – first Eigen value. (for only 2×2)

Rayleigh Quotient Method

- ✓ Starting with an arbitrary vector x_0 we form the sequence of vectors.
- ✓ $x_1 = Ax_0\,, x_2 = Ax_1\,, \dots, x_{k+1} = Ax_k$.
- ✓ Obtain the Rayleigh quotients by $q_k = \frac{x_k' \, x_{k+1}'}{x_k' \cdot x_k'}$.

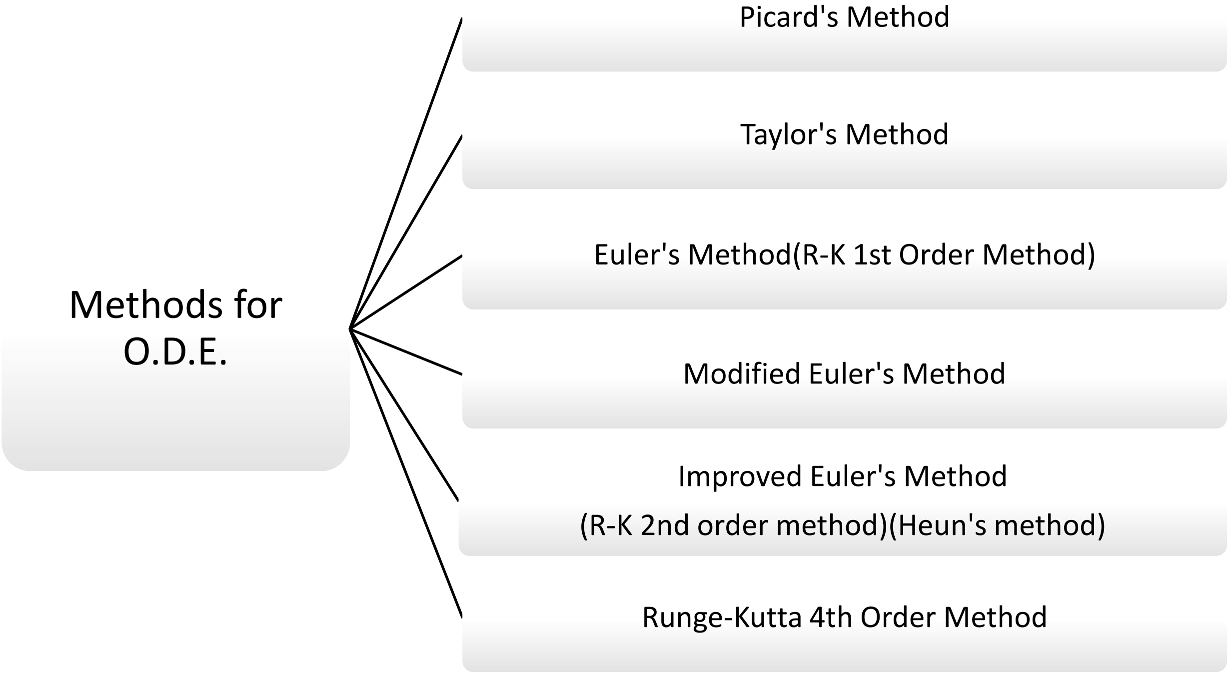
Note

- ✓ Rayleigh quotient method is used to find the dominant eigenvalue of a real symmetric matrix.
- ✓ The q_k are scalars, Since it is the ratio of two scalar products.

Exercise-4

H	Que.1	Use power method to find the largest of Eigen values of the matrix $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$. Perform four iterations only. $[\mathbf{4.91}]$	Nov-10
C	Que.2	Determine the largest eigenvalues of matrix of $A = \begin{bmatrix} -1 & 1 & 4 \\ 10 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ by power method. $[\,\mathbf{5.78}\,]$	Jun-12
H	Que.3	Find the largest eigen value of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ by power method. $[\,\mathbf{3.4142}\,]$	May-15
H	Que.4	Determine the largest eigenvalues of matrix of $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ by power method. $[\,\mathbf{4}\,]$	Dec-15
H	Que.5	Determine the largest eigenvalues of matrix of $A = \begin{bmatrix} -1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ by power method. $[\,\mathbf{7.184}]$	Dec-15

H	Que.6	Determine the largest eigen value of $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ by Power method. $[\mathbf{6.98}]$	Jun-14
C	Que.7	Choosing $x_0 = [1,1,1]^T$writing $x_{i+1} = Ax_i$ & assigning $x_3 = x, x_4 = y$. Apply the power method to find Eigen value of matrix A, compute the Rayleigh quotient & an error bound at this stage, where $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. $[\mathbf{4.99, 4.97, 0.02}]$	Dec-12
H	Que.8	By Rayleigh quotient method find the dominant Eigen value of $A = \begin{bmatrix} 5 & 4 \\ 4 & 3 \end{bmatrix}$. $[\mathbf{8.12}]$	
H	Que.9	Find the dominant Eigen value of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by power method and hence find the other Eigen value also. Verify your results by any other matrix theory. $[\mathbf{5.38, -0.38, 5.37}]$	Jun-10
C	Que.10	By Rayleigh quotient method find the dominant Eigen value of $A = \begin{bmatrix} 10 & 7 & 8 \\ 7 & 5 & 6 \\ 8 & 6 & 10 \end{bmatrix}$. $[\mathbf{22.76}]$	
H	Que.11	Use power method to find the largest of Eigen values of the matrix $A = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$. $[\mathbf{6.7}]$	Jun-11
H	Que.12	Find the dominant Eigen value of $A = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$ by power method. $[\mathbf{7.00}]$	Nov-11
C	Que.13	Find numerically smallest Eigen value of the given matrix using power method, correct up to three decimal places. $\begin{bmatrix} -15 & 4 & 3 \\ 10 & -12 & 6 \\ 20 & -4 & 2 \end{bmatrix}$. $[\mathbf{largest\ eigen\ value = 0.20, smallest\ eigen\ value = 5}]$	Dec-14



SR. NO.	TOPIC NAME
1	Picard's Method
2	Taylor 's Method
3	Euler's Method (R-K 1st Order Method)
4	Modified Euler's Method
5	Improved Euler's Method (Heun's Method OR Runge-Kutta 2nd Order Method)
6	Runge-Kutta 4th Order Method

Picard's Method

- ✓ If $\frac{dy}{dx} = f(x, y); y(x_0) = y_0$
- ✓ Picard's formula

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx \; ; n = 1,2,3, \ldots$$

Note

- ✓ We stop the process, When $y_n = y_{n-1}$, up to the desired decimal places.
- ✓ This method is applicable only to a limited class of equations in which successive integrations can be performed easily.

Exercise-1

C	**Que 1.**	Using Picard's method solve $\frac{dy}{dx} - 1 = xy$ with initial condition $y(0) = 1$, compute $y(0.1)$ correct to three decimal places. $[\mathbf{y(0.1) = 1.105}]$	
T	**Que 2.**	Solve $\frac{dy}{dx} = 3 + 2xy$.Where $y(0) = 1$, for $x = 0.1$ by Picard's method. $[\mathbf{y(0.1) = 1.3121}]$	**Jun-12**
T	**Que 3.**	Obtain Picard's second approximation solution of the initial value problem $\frac{dy}{dx} = x^2 + y^2$ for $x = 0.4$ correct places, given that $y(0) = 0$. $[\mathbf{y(0.4) = 0.0214}]$	
H	**Que 4.**	Using Picard's method solve $\frac{dy}{dx} = x + y^2, y(0) = 1$. $\left[\mathbf{y_2 = 1 + x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{x^4}{4} + \frac{x^5}{20}}\right]$	

Taylor Series

- ✓ If $\frac{dy}{dx} = f(x, y); y(x_0) = y_0$
- ✓ Taylor's series expansion

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!}\, y'(x_0) + \frac{(x - x_0)^2}{2!}\, y''(x_0) + \cdots$$

Putting $x = x_1 = x_0 + h \Rightarrow\ x - x_0 = h$

$$\therefore\ y(x_1) = y(x_0 + h) =\ y(x_0) + \frac{h}{1!}\, y'(x_0) + \frac{h^2}{2!}\, y''(x_0) + \cdots$$

So, $y(x_1) = y_1 = y_0 + \frac{h}{1!}\, y_0{}' + \frac{h^2}{2!} y_0{}'' + \cdots$

$$\therefore\ y(x_n) = y_n = y_{n-1} + \frac{h}{1!}\, y'_{n-1} + \frac{h^2}{2!}\, y''_{n-1} + \frac{h^3}{3!}\, y'''_{n-1} + \cdots$$

Where, $h = x_n - x_{n-1}\,; n = 1,2,3, \ldots$

Exercise-2

C	**Que 1.**	Use Taylor's series method to solve $\frac{dy}{dx} = x^2y - 1, y(0) = 1$. Also find $y(0.03)$. $[\mathbf{y(0.03) = 0.9700}]$	**Dec-10**
T	**Que 2.**	Using Taylor series method, find correct four decimal place, the value of $y(0.1)$, given $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 1$. $[\mathbf{y(0.1) = 1.1111}]$	**Jun-11**
C	**Que 3.**	Solve the Ricatti equation $y' = x^2 + y^2$ using the Taylor's series method for the initial condition $y(0) = 0$. Where $0 \le x \le 0.4$ and $h = 0.2$. $[\mathbf{y(0.2) = 0.0027, y(0.4) = 0.0214}]$	
H	**Que 4.**	Using Taylor series method, find $y(0.1)$ correct to four decimal places, if $y(x)$ satisfies $\frac{dy}{dx} = x - y^2\,, y(0) = 1$. $[\mathbf{y(0.1) = 0.9138}]$	
H	**Que 5.**	Evaluate $y(0.1)$ correct to four decimal places using Taylor's series method if $\frac{dy}{dx} = y^2 + x\,, y(0) = 1$. $[\mathbf{y(0.1) = 1.116}]$	**May-15**
T	**Que 6.**	Using Taylor's series method ,find $y(1.1)$ correct to four decimal places, given that $\frac{dy}{dx} = xy^{\frac{1}{3}}, y(1) = 1$. $[\mathbf{y(1.1) = 1.1068}]$	**Dec-15**

Euler's Method (RK 1st order method)

- ✓ If $\frac{dy}{dx} = f(x, y); y(x_0) = y_0$
- ✓ Euler's Formula

$$y_{n+1} = y_n + h\, f(x_n, y_n); n = 0,1,2, \ldots$$

Where, $h = x_n - x_{n-1}\,; n = 1,2,3, \ldots$

Explanation:

Let $[a, b]$ be the interval over which we want to find the solution of

$$\frac{dy}{dx} = f(x, y)\ ; a < x < b\ ; y(x_0) = y_0 \ldots (1)$$

A set of points $\{(x_n, y_n)\}$ are generated which are used for an approximation $[\text{i.e. } y(x_n) = y_n]$

For convenience, we divide $[a, b]$ into n equal subintervals.

$\Rightarrow x_n = x_0 + nh\ ; n = 0,1,2, \ldots\ ;$ where, $h = \frac{b-a}{n}$ is called the step size.

Now, $y(x)$ is expand by using Taylor's series about $x = x_0$ as following

$$y(x) = y(x_0) + \frac{(x - x_0)}{1\,!} y'(x_0) + \frac{(x - x_0)^2}{2\,!} y''(x_0) + \frac{(x - x_0)^3}{3\,!} y'''(x_0) + \cdots \ldots (2)$$

We have, $\left[\frac{dy}{dx}\right]_{x=x_0} = y'(x_0) = f(x_0, y_0) = f\big(x_0, y(x_0)\big)$

By Eq. (2),

$$y(x_1) = y(x_0) + \frac{(x_1 - x_0)}{1\,!} y'(x_0) + \frac{(x_1 - x_0)^2}{2\,!} y''(x_0) + \frac{(x_1 - x_0)^3}{3\,!} y'''(x_0) + \cdots$$

Take $h = x_1 - x_0$

$$y(x_1) = y(x_0) + \frac{h}{1\,!} f\big(x_0, y(x_0)\big) + \frac{(x_1 - x_0)^2}{2\,!} y''(x_0) + \frac{(x_1 - x_0)^3}{3\,!} y'''(x_0) + \cdots$$

If the step size is chosen too small enough, then we may neglect the second order term involving h^2 and get

$$y_1 = y(x_1) = y(x_0) + \frac{h}{1\,!} f\big(x_0, y(x_0)\big) = y_0 + hf(x_0, y_0)$$

Which is called Euler's Approximation.

The process is repeated and generates a sequence of points that approximate the solution curve $y = y(x)$.

The general step for Euler's method is $y_{n+1} = y_n + h\, f(x_n, y_n)\ ; n = 0,1,2, \ldots$

Exercise-3

C	Que 1.	Describe Euler's Method for first order ordinary differential equation.	**Dec-10** **Jun-12**
C	Que 2.	Apply Euler's method to find the approximate solution of $\frac{dy}{dx} = x + y$ with $y(0) = 0$ and $h = 2$. Show your calculation up to five iteration. $[\mathbf{y_5 = 232}]$	**Jun-13**

T	Que 3.	Derive Euler's formula for initial value problem $\frac{dy}{dx} = f(x, y); y(x_0) = y_0$. Hence , use it find the value of y for $\frac{dy}{dx} = x + y; y(0) = 1$ when $x = 0.1, 0.2$ with step size $h = 0.05$. Also Compare with analytic solution. $[\boldsymbol{Y(0.1) = 1.1050, Y(0.2) = 1.2311}]$	May-15
T	Que 4.	Apply Euler's method to solve the initial value problem $\frac{dy}{dx} = x + y$, with $y(0) = 0$ with choosing $h = 0.2$ and compute $y_1, y_2, y_3 \ldots \ldots y_5$ Compare your result with the exact solution. $[\mathbf{y_1 = 1, y_2 = 0.04, y_3 = 0.128, y_4 = 0.236, y_5 = 0.4883}]$	
H	Que 5.	Using Euler's method ,find an approximate value of y corresponding to $x = 1$ given that $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$. $[\mathbf{For\ h = 0.25, y_4 = 2.8828}]$	Jun -13
T	Que 6.	Use Euler method to find y(1.4) given that $\frac{dy}{dx} = xy^{\frac{1}{2}}$, $y(1) = 1$. $[\mathbf{y(1.4) = 1.4986}]$	Dec-10
T	Que 7.	Use Euler method to find y(0.2) given that $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$. (Take $h = 0.1$) $[\mathbf{y(0.2) = 1.1918}]$	Jun-11
H	Que 8.	Use Euler method to obtain an approximate value of y(0.4) for the equation $\frac{dy}{dx} = x + y$, $y(0) = 1$ with $h = 0.1$. $[\mathbf{y(0.4) = 1.5282}]$	Dec-11
C	Que 9.	Use the Euler's method, find y(0.04) for the following initial value problem.$\frac{dy}{dx} = y$, $y(0) = 1$.Take first step size as $h = 0.01$. $[\mathbf{y(0.04) = 1.0406}]$	
H	Que 10.	Explain Euler's method for solving first order ordinary differential equation. Hence use this method, find y(2) for $\frac{dy}{dx} = x + 2y$ with $y(1) = 1$. $[\mathbf{y(2) = 5.75}]$	Dec-15
T	Que 11.	Solve initial value problem $\frac{dy}{dx} = x\sqrt{y}$, $y(1) = 1$ and hence find y(1.5) by taking $h = 0.1$ using Euler's method. $[\mathbf{y(1.5) = 1.6815}]$	May-15
C	Que 12.	Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with initial condition $y = 1$ at $x = 0$; find y for $x = 1$ and $h = 0.25$ by Euler's method. $[\mathbf{y(1) = 1.6227}]$	Dec-15
H	Que 13.	Use Euler's method to find an approximation value of y at $x = 0.1$ for the initial value problem $\frac{dy}{dx} = x - y^2; y(0) = 1$. $[\mathbf{y(0.1) = 0.9133}]$	Dec-15

Modified Euler's Method

- ✓ If $\frac{dy}{dx} = f(x, y); y(x_0) = y_0$
- ✓ Modified Euler's Formula

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right); n = 0,1,2, \dots$$

Where, $h = x_n - x_{n-1}$; $n = 1,2,3, \dots$

Exercise-4

C	Que 1.	Using modified Euler's method solve $\frac{dy}{dx} = x^2 + y$ with the initial condition $y(0) = 1$and compute $y(0.02), y(0.04)$. compare the answer with exact solution. <table><tr><td></td><td>$\mathbf{y(0.02)}$</td><td>$\mathbf{y(0.04)}$</td></tr><tr><td>**Modified Euler's**</td><td>$\mathbf{1.0202}$</td><td>$\mathbf{1.0408}$</td></tr><tr><td>**Exact solution**</td><td>$\mathbf{1.0202}$</td><td>$\mathbf{1.0408}$</td></tr></table>	
H	Que 2.	Using modified Euler's method solve $\frac{dy}{dx} = y - \frac{2x}{y}$ with the initial condition $y(0) = 1$ and compute $y(0.2)$, taking $h = 0.1$. $[\mathbf{y(0.2) = 1.1833}]$	
T	Que 3.	Using modified Euler's method to obtain $y(0.2), y(0.4)$ and $y(0.6)$ correct to three decimal places given that $\frac{dy}{dx} = y - x^2$ with initial condition $y(0) = 1$. $[\mathbf{y(0.2) = 1.218, y(0.4) = 1.467, y(0.6) = 1.737}]$	

Improved Euler's Method (Heun's Method / R-K 2nd Order Method)

- ✓ If $\frac{dy}{dx}=f(x,y);y(x_0)=y_0$
- ✓ Improved Euler's Formula

$$y_{n+1}=y_n+\frac{h}{2}[f(x_n,y_n)+f(x_n+h,y_n+hf(x_n,y_n))];n=0,1,2,\dots$$

Where, $h=x_n-x_{n-1}$; $n=1,2,3,\dots$

Exercise-5

<table>
<tr><td>C</td><td>Que 1.</td><td>Using improved Euler's method solves $\frac{dy}{dx}=1-y$ with the initial condition $y(0)=0$ and tabulates the solution at $x=0.1$, 0.2. Compare the answer with exact solution.
<table>
<tr><th></th><th>$\mathbf{y(0.1)}$</th><th>$\mathbf{y(0.2)}$</th></tr>
<tr><td>Improved Euler's</td><td>$\mathbf{0.0950}$</td><td>$\mathbf{0.1810}$</td></tr>
<tr><td>Exact solution</td><td>$\mathbf{0.0952}$</td><td>$\mathbf{0.1813}$</td></tr>
</table></td><td>Nov-11</td></tr>
<tr><td>H</td><td>Que 2.</td><td>Using improved Euler's method solves $\frac{dy}{dx}+2xy^2=0$ with the initial condition $y(0)=1$ and compute $y(1)$ taking $h=0.2$ compare the answer with exact solution.
<table>
<tr><th></th><th>$\mathbf{y(0)}$</th><th>$\mathbf{y(0.2)}$</th><th>$\mathbf{y(0.4)}$</th><th>$\mathbf{y(0.6)}$</th><th>$\mathbf{y(0.8)}$</th><th>$\mathbf{y(1)}$</th></tr>
<tr><td>Improved Euler's</td><td>$\mathbf{1}$</td><td>$\mathbf{0.9600}$</td><td>$\mathbf{0.8603}$</td><td>$\mathbf{0.7350}$</td><td>$\mathbf{0.6115}$</td><td>$\mathbf{0.5033}$</td></tr>
<tr><td>Exact solution</td><td>$\mathbf{1}$</td><td>$\mathbf{0.9615}$</td><td>$\mathbf{0.8621}$</td><td>$\mathbf{0.7353}$</td><td>$\mathbf{0.6098}$</td><td>$\mathbf{0.5000}$</td></tr>
</table></td><td>Jun-10</td></tr>
<tr><td>T</td><td>Que 3.</td><td>Given the equation $\frac{dy}{dx}=\frac{2y}{x}$; $y(1)=2$ Estimate $y(2)$ using Heun's method $h=0.25$ and compare the results with exact answers.
<table>
<tr><th></th><th>$\mathbf{y(1)}$</th><th>$\mathbf{y(1.25)}$</th><th>$\mathbf{y(1.50)}$</th><th>$\mathbf{y(1.75)}$</th><th>$\mathbf{y(2)}$</th></tr>
<tr><td>Improved Euler's</td><td>$\mathbf{2}$</td><td>$\mathbf{3.1000}$</td><td>$\mathbf{4.4433}$</td><td>$\mathbf{6.0302}$</td><td>$\mathbf{7.8608}$</td></tr>
<tr><td>Exact solution</td><td>$\mathbf{2}$</td><td>$\mathbf{3.1250}$</td><td>$\mathbf{4.5000}$</td><td>$\mathbf{6.1250}$</td><td>$\mathbf{8.0000}$</td></tr>
</table></td><td></td></tr>
<tr><td>T</td><td>Que 4.</td><td>Apply improved Euler method to solve the initial value problem $y'=x+y$ with $y(0)=0$ choosing $h=0.2$ and compute y_1,y_2,y_3,y_4,y_5.Compare your results with the exact solutions.
$[\mathbf{y_1=0.02,y_2=0.0884,y_3=0.2158,y_4=0.153,y_5=0.7027}]$</td><td>Dec-14</td></tr>
<tr><td>C</td><td>Que 5.</td><td>Use Runge-Kutta second order method to find the approximate value of $y(0.2)$ given that $\frac{dy}{dx}=x-y^2$ and $y(0)=1$ and $h=0.1$.
$[\mathbf{y(0.2)=0.8523}]$</td><td>Dec-10
Dec-15</td></tr>
<tr><td>H</td><td>Que 6.</td><td>Given that $y=1.3$ when $x=1$ and $\frac{dy}{dx}=3x+y$ use second order RK method (i.e. Heun's method) to approximate y ,when $x=1.2$ use step size 0.1.
$[\mathbf{y(1.2)=2.3135}]$</td><td>Dec-12</td></tr>
</table>

Runge Kutta 4th Order Method

- ✓ If $\frac{dy}{dx} = f(x, y); y(x_0) = y_0$
- ✓ RK 4th Order Formula $y_{n+1} = y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$$x_{n+1} = x_n + h\ ; n = 0,1,2, \dots$$

Where,

$$K_1 = h\,f(x_n, y_n)$$

$$K_2 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right)$$

$$K_3 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{K_2}{2}\right)$$

$$K_4 = h\,f(x_n + h, y_n + K_3)$$

Exercise-6

C	Que 1.	Write formula for Runge-Kutta method for order four.	Jun-13
H	Que 2.	a)Use the fourth-order Runge-Kutta to solve $10\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$.Evaluate the value of y when $x = 0.1$. $[\mathbf{y(0.1) = 1.0101}]$ (b)Given $10\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$.Using fourth –order Runge-Kutta method. Find y(0.2) & y(0.4) with $h = 0.1$. $[\mathbf{y(0.2) = 1.0206, y(0.4) = 1.0438}]$	Dec-11 May -15
H	Que 3.	Apply Runge-Kutta method of fourth order to calculate y(0.2) given $\frac{dy}{dx} = x + y$, $y(0) = 1$ taking $h = 0.1$ $[\mathbf{y(0.2) = 1.2428}]$	Jun-10 Jun-11 Dec-12
T	Que 4.	Use Runge–Kutta fourth order method to find y(1.1) given that $\frac{dy}{dx} = x - y$, $y(1) = 1$and $h = 0.05$. $[\mathbf{y(1.1) = 1.0053}]$	Dec-10
T	Que 5.	Describe y(0.1) and y(0.2) Correct to four decimal places from $\frac{dy}{dx} = 2x + y$, $y(0) = 1$ use fourth order R-K method. $[\mathbf{y(0.1) = 1.1155, y(0.2) = 1.2642}]$	Jun-12
C	Que 6.	Apply Runge-Kutta fourth order method, to find an approximate value of ywhen $x = 0.2$ in steps of 0.1, if $\frac{dy}{dx} = x + y^2$, given that $y = 1$ when $x = 0$. $[\mathbf{y(0.1) = 1.1165, y(0.2) = 1.2736}]$	Jun -14

C	Que 7.	Using the Range-Kutta method of fourth order , find y at $x = 0.1$ given diff. equation $\frac{dy}{dx} = 3e^x + 2y, y(0) = 0$ by taking $h = 0.1$ and also compare the solution with exact solution. | | $\mathbf{y(0.1)}$ | | **R-K method** | $\mathbf{0.3487}$ | | **Exact solution** | $\mathbf{0.348}$ |	May -15
T	Que 8.	Apply Runge-Kutta fourth order method to calculate y(0.2) and y(0.4) given $\frac{dy}{dx} = y - \frac{2x}{y}, y(0) = 1$. $[\mathbf{y(0.2) = 1.1832, y(0.4) = 1.3416}]$	Dec-15
H	Que 9.	Solve initial value problem $\frac{dy}{dx} = -2xy^2; y(0) = 1$ with $h = 0.2$ for y(0.2) using Runge-Kutta fourth order method. $[\mathbf{y(0.2) = 0.96153}]$	Dec-15